Chapter 5
Fields of Force

“Okay. Your duties are as follows: Get Breen. I don’t care how you get him, but get him soon. That faker! He posed for twenty years as a scientist without ever being apprehended. Well, I’m going to do some apprehending that’ll make all previous apprehending look like no apprehension at all. You with me?”

“Yes,” said Battle, very much confused. “What’s that thing you have?”

“Piggy-back heat-ray. You transpose the air in its path into an unstable isotope which tends to carry all energy as heat. Then you shoot your juice light, or whatever along the isotopic path and you burn whatever’s on the receiving end. You want a few?”

“No,” said Battle. “I have my gats. What else have you got for offense and defense?” Underbottam opened a cabinet and proudly waved an arm. “Everything,” he said.

“Disintegraters, heat-rays, bombs of every type. And impenetrable shields of energy, massive and portable. What more do I need?”

From THE REVERSIBLE REVOLUTIONS by Cecil Corwin, Cosmic Stories, March 1941. Art by Morey, Bok, Kyle, Hunt, Forte. Copyright expired.

Cutting-edge science readily infiltrates popular culture, though sometimes in garbled form. The Newtonian imagination populated the universe mostly with that nice solid stuff called matter, which was made of little hard balls called atoms. In the early twentieth century, consumers of pulp fiction and popularized science began to hear of a new image of the universe, full of x-rays, N-rays, and Hertzian waves. What they were beginning to soak up through their skins was a drastic revision of Newton’s concept of a universe made of chunks of matter which happened to interact via forces. In the newly emerging picture, the universe was made of force, or, to be more technically accurate, of ripples in universal fields of force. Unlike the average reader of Cosmic Stories in 1941, you now possess enough technical background to understand what a “force field” really is.

5.1 Why Fields?
Time delays in forces exerted at a distance

What convinced physicists that they needed this new concept of a field of force? Although we have been dealing mostly with elec-
A bar magnet's atoms are (partially) aligned. A bar magnet interacts with our magnetic planet. Magnets aligned north-south.

Trical forces, let’s start with a magnetic example. (In fact the main reason I’ve delayed a detailed discussion of magnetism for so long is that mathematical calculations of magnetic effects are handled much more easily with the concept of a field of force.) First a little background leading up to our example. A bar magnet, a, has an axis about which many of the electrons’ orbits are oriented. The earth itself is also a magnet, although not a bar-shaped one. The interaction between the earth-magnet and the bar magnet, b, makes them want to line up their axes in opposing directions (in other words such that their electrons rotate in parallel planes, but with one set rotating clockwise and the other counterclockwise as seen looking along the axes). On a smaller scale, any two bar magnets placed near each other will try to align themselves head-to-tail, c.

Now we get to the relevant example. It is clear that two people separated by a paper-thin wall could use a pair of bar magnets to signal to each other. Each person would feel her own magnet trying to twist around in response to any rotation performed by the other person’s magnet. The practical range of communication would be very short for this setup, but a sensitive electrical apparatus could pick up magnetic signals from much farther away. In fact, this is not so different from what a radio does: the electrons racing up and down the transmitting antenna create forces on the electrons in the distant receiving antenna. (Both magnetic and electric forces are involved in real radio signals, but we don’t need to worry about that yet.)

A question now naturally arises as to whether there is any time delay in this kind of communication via magnetic (and electric) forces. Newton would have thought not, since he conceived of physics in terms of instantaneous action at a distance. We now know, however, that there is such a time delay. If you make a long-distance phone call that is routed through a communications satellite, you should easily be able to detect a delay of about half a second over the signal’s round trip of 50,000 miles. Modern measurements have shown that electric, magnetic, and gravitational forces all travel at the speed of light, $3 \times 10^8$ m/s.$^1$ (In fact, we will soon discuss how light itself is made of electricity and magnetism.)

If it takes some time for forces to be transmitted through space, then apparently there is some thing that travels through space. The fact that the phenomenon travels outward at the same speed in all directions strongly evokes wave metaphors such as ripples on a pond.

**More evidence that fields of force are real: they carry energy.**

The smoking-gun argument for this strange notion of traveling force ripples comes from the fact that they carry energy.

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$^1$ As discussed in book 6 of this series, one consequence of Einstein’s theory of relativity is that material objects can never move faster than the speed of light. It can also be shown that signals or information are subject to the same limit.
First suppose that the person holding the bar magnet on the right decides to reverse hers, resulting in configuration d. She had to do mechanical work to twist it, and if she releases the magnet, energy will be released as it flips back to c. She has apparently stored energy by going from c to d. So far everything is easily explained without the concept of a field of force.

But now imagine that the two people start in position c and then simultaneously flip their magnets extremely quickly to position e, keeping them lined up with each other the whole time. Imagine, for the sake of argument, that they can do this so quickly that each magnet is reversed while the force signal from the other is still in transit. (For a more realistic example, we’d have to have two radio antennas, not two magnets, but the magnets are easier to visualize.) During the flipping, each magnet is still feeling the forces arising from the way the other magnet used to be oriented. Even though the two magnets stay aligned during the flip, the time delay causes each person to feel resistance as she twists her magnet around. How can this be? Both of them are apparently doing mechanical work, so they must be storing magnetic energy somehow. But in the traditional Newtonian conception of matter interacting via instantaneous forces at a distance, interaction energy arises from the relative positions of objects that are interacting via forces. If the magnets never changed their orientations relative to each other, how can any magnetic energy have been stored?

The only possible answer is that the energy must have gone into the magnetic force ripples crisscrossing the space between the magnets. Fields of force apparently carry energy across space, which is strong evidence that they are real things.

This is perhaps not as radical an idea to us as it was to our ancestors. We are used to the idea that a radio transmitting antenna consumes a great deal of power, and somehow spews it out into the universe. A person working around such an antenna needs to be careful not to get too close to it, since all that energy can easily cook flesh (a painful phenomenon known as an “RF burn”).

### 5.2 The Gravitational Field

Given that fields of force are real, how do we define, measure, and calculate them? A fruitful metaphor will be the wind patterns experienced by a sailing ship. Wherever the ship goes, it will feel a certain amount of force from the wind, and that force will be in a certain direction. The weather is ever-changing, of course, but for now let’s just imagine steady wind patterns. Definitions in physics are operational, i.e., they describe how to measure the thing being defined. The ship’s captain can measure the wind’s “field of force” by going to the location of interest and determining both the direction of the wind and the strength with which it is blowing. Charting
all these measurements on a map leads to a depiction of the field of wind force like the one shown in the figure. This is known as the “sea of arrows” method of visualizing a field.

Now let’s see how these concepts are applied to the fundamental force fields of the universe. We’ll start with the gravitational field, which is the easiest to understand. As with the wind patterns, we’ll start by imagining gravity as a static field, even though the existence of the tides proves that there are continual changes in the gravity field in our region of space. Defining the direction of the gravitational field is easy enough: we simply go to the location of interest and measure the direction of the gravitational force on an object, such as a weight tied to the end of a string.

But how should we define the strength of the gravitational field? Gravitational forces are weaker on the moon than on the earth, but we cannot specify the strength of gravity simply by giving a certain number of newtons. The number of newtons of gravitational force depends not just on the strength of the local gravitational field but also on the mass of the object on which we’re testing gravity, our “test mass.” A boulder on the moon feels a stronger gravitational force than a pebble on the earth. We can get around this problem by defining the strength of the gravitational field as the force acting on an object, divided by the object’s mass.

**definition of the gravitational field**

The gravitational field vector, \( \mathbf{g} \), at any location in space is found by placing a test mass \( m_t \) at that point. The field vector is then given by \( \mathbf{g} = \mathbf{F}/m_t \), where \( \mathbf{F} \) is the gravitational force on the test mass.

The magnitude of the gravitational field near the surface of the earth is about 9.8 N/kg, and it’s no coincidence that this number looks familiar, or that the symbol \( \mathbf{g} \) is the same as the one for gravitational acceleration. The force of gravity on a test mass will equal \( m_t \mathbf{g} \), where \( \mathbf{g} \) is the gravitational acceleration. Dividing by \( m_t \) simply gives the gravitational acceleration. Why define a new name and new units for the same old quantity? The main reason is that it prepares us with the right approach for defining other fields.

The most subtle point about all this is that the gravitational field tells us about what forces would be exerted on a test mass by the earth, sun, moon, and the rest of the universe, if we inserted a test mass at the point in question. The field still exists at all the places where we didn’t measure it.

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**Gravitational field of the earth example 1**

- What is the magnitude of the earth’s gravitational field, in terms of its mass, \( M \), and the distance \( r \) from its center?

- Substituting \( |\mathbf{F}| = G M m_t / r^2 \) into the definition of the gravitational field, we find \( |\mathbf{g}| = G M / r^2 \). This expression could be used for
the field of any spherically symmetric mass distribution, since the
equation we assumed for the gravitational force would apply in
any such case.

Sources and sinks

If we make a sea-of-arrows picture of the gravitational fields
surrounding the earth, $g$, the result is evocative of water going down
a drain. For this reason, anything that creates an inward-pointing
field around itself is called a sink. The earth is a gravitational sink.
The term “source” can refer specifically to things that make outward
fields, or it can be used as a more general term for both “outies”
and “innies.” However confusing the terminology, we know that
gravitational fields are only attractive, so we will never find a region
of space with an outward-pointing field pattern.

Knowledge of the field is interchangeable with knowledge of its
sources (at least in the case of a static, unchanging field). If aliens
saw the earth’s gravitational field pattern they could immediately
infer the existence of the planet, and conversely if they knew the
mass of the earth they could predict its influence on the surrounding
gravitational field.

Superposition of fields

A very important fact about all fields of force is that when there
is more than one source (or sink), the fields add according to the
rules of vector addition. The gravitational field certainly will have
this property, since it is defined in terms of the force on a test
mass, and forces add like vectors. Superposition is an important
characteristics of waves, so the superposition property of fields is
consistent with the idea that disturbances can propagate outward
as waves in a field.

Reduction in gravity on Io due to Jupiter’s gravity

Example 2

The average gravitational field on Jupiter’s moon Io is 1.81 N/kg.
By how much is this reduced when Jupiter is directly overhead?
Io’s orbit has a radius of $4.22 \times 10^8$ m, and Jupiter’s mass is
$1.899 \times 10^{27}$ kg.

By the shell theorem, we can treat the Jupiter as if its mass was
all concentrated at its center, and likewise for Io. If we visit Io and
land at the point where Jupiter is overhead, we are on the same
line as these two centers, so the whole problem can be treated
one-dimensionally, and vector addition is just like scalar addition.
Let’s use positive numbers for downward fields (toward the center
of Io) and negative for upward ones. Plugging the appropriate
data into the expression derived in example 1, we find that the
Jupiter’s contribution to the field is $-0.71$ N/kg. Superposition
says that we can find the actual gravitational field by adding up
the fields created by Io and Jupiter: $1.81 - 0.71 = 1.1$ N/kg.
You might think that this reduction would create some spectacular

\[ g / The \ gravitational \ field \ surrounding \ a \ clump \ of \ mass \ such \ as \ the \ earth. \]

\[ h / The \ gravitational \ fields \ of \ the \ earth \ and \ moon \ superpose. \ Note \ how \ the \ fields \ cancel \ at \ one \ point, \ and \ how \ there \ is \ no \ boundary \ between \ the \ interpenetrating \ fields \ surrounding \ the \ two \ bodies. \]
effects, and make Io an exciting tourist destination. Actually you would not detect any difference if you flew from one side of Io to the other. This is because your body and Io both experience Jupiter’s gravity, so you follow the same orbital curve through the space around Jupiter.

**Gravitational waves**

A source that sits still will create a static field pattern, like a steel ball sitting peacefully on a sheet of rubber. A moving source will create a spreading wave pattern in the field, like a bug thrashing on the surface of a pond. Although we have started with the gravitational field as the simplest example of a static field, stars and planets do more stately gliding than thrashing, so gravitational waves are not easy to detect. Newton’s theory of gravity does not describe gravitational waves, but they are predicted by Einstein’s general theory of relativity. J.H. Taylor and R.A. Hulse were awarded the Nobel Prize in 1993 for giving indirect evidence that Einstein’s waves actually exist. They discovered a pair of exotic, ultra-dense stars called neutron stars orbiting one another very closely, and showed that they were losing orbital energy at the rate predicted by Einstein’s theory.

A Caltech-MIT collaboration has built a pair of gravitational wave detectors called LIGO to search for more direct evidence of gravitational waves. Since they are essentially the most sensitive vibration detectors ever made, they are located in quiet rural areas, and signals will be compared between them to make sure that they were not due to passing trucks. The project began operating at full
sensitivity in 2005, and is now able to detect a vibration that causes a change of $10^{-18}$ m in the distance between the mirrors at the ends of the 4-km vacuum tunnels. This is a thousand times less than the size of an atomic nucleus! There is only enough funding to keep the detectors operating for a few more years, so the physicists can only hope that during that time, somewhere in the universe, a sufficiently violent cataclysm will occur to make a detectable gravitational wave. (More accurately, they want the wave to arrive in our solar system during that time, although it will have been produced millions of years before.)

5.3 The Electric Field

Definition

The definition of the electric field is directly analogous to, and has the same motivation as, the definition of the gravitational field:

**definition of the electric field**

The electric field vector, $\mathbf{E}$, at any location in space is found by placing a test charge $q_t$ at that point. The electric field vector is then given by $\mathbf{E} = \mathbf{F}/q_t$, where $\mathbf{F}$ is the electric force on the test charge.

Charges are what create electric fields. Unlike gravity, which is always attractive, electricity displays both attraction and repulsion. A positive charge is a source of electric fields, and a negative one is a sink.

The most difficult point about the definition of the electric field is that the force on a negative charge is in the opposite direction compared to the field. This follows from the definition, since dividing a vector by a negative number reverses its direction. It’s as though we had some objects that fell upward instead of down.

**self-check A**

Find an equation for the magnitude of the field of a single point charge $Q$.

Answer, p. 205

Superposition of electric fields

Charges $q$ and $-q$ are at a distance $b$ from each other, as shown in the figure. What is the electric field at the point $P$, which lies at a third corner of the square?

The field at $P$ is the vector sum of the fields that would have been created by the two charges independently. Let positive $x$ be to the right and let positive $y$ be up.

Negative charges have fields that point at them, so the charge $-q$ makes a field that points to the right, i.e., has a positive $x$
A dipole field. Electric fields diverge from a positive charge and converge on a negative charge.

A water molecule is a dipole. The electrons tend to shift away from the hydrogen atoms and onto the oxygen atom.

Dipoles

The simplest set of sources that can occur with electricity but not with gravity is the dipole, consisting of a positive charge and a negative charge with equal magnitudes. More generally, an electric dipole can be any object with an imbalance of positive charge on one side and negative on the other. A water molecule, l, is a dipole because the electrons tend to shift away from the hydrogen atoms and onto the oxygen atom.

Your microwave oven acts on water molecules with electric fields. Let us imagine what happens if we start with a uniform electric field, \( E \), made by some external charges, and then insert a dipole, \( m/2 \), consisting of two charges connected by a rigid rod. The dipole disturbs the field pattern, but more important for our present purposes is that it experiences a torque. In this example, the positive charge

\[
E_{q,x} = \frac{kq}{b^2} \\
E_{q,y} = 0
\]

Note that if we had blindly ignored the absolute value signs and plugged in \(-q\) to the equation, we would have incorrectly concluded that the field went to the left.

By the Pythagorean theorem, the positive charge is at a distance \( \sqrt{2}b \) from \( P \), so the magnitude of its contribution to the field is \( E = kq/2b^2 \). Positive charges have fields that point away from them, so the field vector is at an angle of 135° counterclockwise from the \( x \) axis.

\[
E_{q,x} = \frac{kq}{2b^2} \cos 135° = -\frac{kq}{2^{3/2}b^2} \\
E_{q,y} = \frac{kq}{2b^2} \sin 135° = \frac{kq}{2^{3/2}b^2}
\]

The total field is

\[
E_x = \left(1 - 2^{-3/2}\right) \frac{kq}{b^2} \\
E_y = \frac{kq}{2^{3/2}b^2}
\]
feels an upward force, but the negative charge is pulled down. The result is that the dipole wants to align itself with the field, \( \mathbf{m}/3 \). The microwave oven heats food with electrical (and magnetic) waves. The alternation of the torque causes the molecules to wiggle and increase the amount of random motion. The slightly vague definition of a dipole given above can be improved by saying that a dipole is any object that experiences a torque in an electric field.

What determines the torque on a dipole placed in an externally created field? Torque depends on the force, the distance from the axis at which the force is applied, and the angle between the force and the line from the axis to the point of application. Let a dipole consisting of charges \( +q \) and \( -q \) separated by a distance \( \ell \) be placed in an external field of magnitude \( |\mathbf{E}| \), at an angle \( \theta \) with respect to the field. The total torque on the dipole is

\[
\tau = \frac{\ell q |\mathbf{E}| \sin \theta}{2} + \frac{\ell q |\mathbf{E}| \sin \theta}{2} = \ell q |\mathbf{E}| \sin \theta.
\]

(Note that even though the two forces are in opposite directions, the torques do not cancel, because they are both trying to twist the dipole in the same direction.) The quantity \( \ell q \) is called the dipole moment, notated \( D \). (More complex dipoles can also be assigned a dipole moment — they are defined as having the same dipole moment as the two-charge dipole that would experience the same torque.)

Dipole moment of a molecule of NaCl gas example 4

In a molecule of NaCl gas, the center-to-center distance between the two atoms is about 0.6 nm. Assuming that the chlorine completely steals one of the sodium’s electrons, compute the magnitude of this molecule’s dipole moment.

The total charge is zero, so it doesn’t matter where we choose the origin of our coordinate system. For convenience, let’s choose it to be at one of the atoms, so that the charge on that atom doesn’t contribute to the dipole moment. The magnitude of the

1. A uniform electric field created by some charges “off-stage.”
2. A dipole is placed in the field.
3. The dipole aligns with the field.
Dipole moment is then

\[ D = (6 \times 10^{-10} \text{ m})(e) \]
\[ = (6 \times 10^{-10} \text{ m})(1.6 \times 10^{-19} \text{ C}) \]
\[ = 1 \times 10^{-28} \text{ C} \cdot \text{m} \]

**Alternative definition of the electric field**

The behavior of a dipole in an externally created field leads us to an alternative definition of the electric field:

**Alternative definition of the electric field**

The electric field vector, \( \mathbf{E} \), at any location in space is defined by observing the torque exerted on a test dipole \( D_t \) placed there. The direction of the field is the direction in which the field tends to align a dipole (from \(-\) to \(+\)), and the field’s magnitude is \( |\mathbf{E}| = \tau / D_t \sin \theta \).

The main reason for introducing a second definition for the same concept is that the magnetic field is most easily defined using a similar approach.

**Voltage related to electric field**

Voltage is potential energy per unit charge, and electric field is force per unit charge. We can therefore relate voltage and field if we start from the relationship between potential energy and force,

\[ \Delta P \mathcal{E} = -Fd \quad , \quad [\text{assuming constant force and motion parallel to the force}] \]

and divide by charge,

\[ \Delta P \mathcal{E} = -Fd \quad , \quad [\text{assuming constant force and motion parallel to the force}] \]

giving

\[ \Delta V = -Ed \quad , \quad [\text{assuming constant force and motion parallel to the force}] \]

In other words, the difference in voltage between two points equals the electric field strength multiplied by the distance between them. The interpretation is that a strong electric field is a region of space where the voltage is rapidly changing. By analogy, a steep hillside is a place on the map where the altitude is rapidly changing.

**Field generated by an electric eel**  
**example 5**

Suppose an electric eel is 1 m long, and generates a voltage difference of 1000 volts between its head and tail. What is the electric field in the water around it?
Example 6. We are only calculating the amount of field, not its direction, so we ignore positive and negative signs. Subject to the possibly inaccurate assumption of a constant field parallel to the eel's body, we have

\[ |E| = \frac{\Delta V}{\Delta x} = 1000 \text{ V/m} \]

The hammerhead shark
One of the reasons hammerhead sharks have their heads shaped the way they do is that, like quite a few other fish, they can sense electric fields as a way of finding prey, which may for example be hidden in the sand. From the equation \( E = \Delta V / \Delta x \), we can see that if the shark is sensing the voltage difference between two points, it will be able to detect smaller electric fields if those two points are farther apart. The shark has a network of sensory organs, called the ampullae of Lorenzini, on the skin of its head. Since the network is spread over a wider head, the \( \Delta x \) is larger. Some sharks can detect electric fields as weak as 50 picovolts per meter!

Relating the units of electric field and voltage
From our original definition of the electric field, we expect it to have units of newtons per coulomb, N/C. The example above, however, came out in volts per meter, V/m. Are these inconsistent? Let's reassure ourselves that this all works. In this kind of situation, the best strategy is usually to simplify the more complex units so that they involve only mks units and coulombs. Since voltage is defined as electrical energy per unit charge, it has units of J/C:

\[ \frac{V}{m} = \frac{J}{C \cdot m} \]

To connect joules to newtons, we recall that work equals force times distance, so \( J = N \cdot m \), so

\[ \frac{V}{m} = \frac{N \cdot m}{C \cdot m} = \frac{N}{C} \]

As with other such difficulties with electrical units, one quickly begins to recognize frequently occurring combinations.
Discussion Questions

A. In the definition of the electric field, does the test charge need to be 1 coulomb? Does it need to be positive?

B. Does a charged particle such as an electron or proton feel a force from its own electric field?

C. Is there an electric field surrounding a wall socket that has nothing plugged into it, or a battery that is just sitting on a table?

D. In a flashlight powered by a battery, which way do the electric fields point? What would the fields be like inside the wires? Inside the filament of the bulb?

E. Criticize the following statement: “An electric field can be represented by a sea of arrows showing how current is flowing.”

F. The field of a point charge, \( |\mathbf{E}| = \frac{kQ}{r^2} \), was derived in the self-check above. How would the field pattern of a uniformly charged sphere compare with the field of a point charge?

G. The interior of a perfect electrical conductor in equilibrium must have zero electric field, since otherwise the free charges within it would be drifting in response to the field, and it would not be in equilibrium. What about the field right at the surface of a perfect conductor? Consider the possibility of a field perpendicular to the surface or parallel to it.

H. Compare the dipole moments of the molecules and molecular ions shown in the figure.

I. Small pieces of paper that have not been electrically prepared in any way can be picked up with a charged object such as a charged piece of tape. In our new terminology, we could describe the tape’s charge as inducing a dipole moment in the paper. Can a similar technique be used to induce not just a dipole moment but a charge?

J. The earth and moon are fairly uneven in size and far apart, like a baseball and a ping-pong ball held in your outstretched arms. Imagine instead a planetary system with the character of a double planet: two planets of equal size, close together. Sketch a sea of arrows diagram of their gravitational field.

5.4 ∫ Voltage for Nonuniform Fields

The calculus-savvy reader will have no difficulty generalizing the field-voltage relationship to the case of a varying field. The potential energy associated with a varying force is

\[
\Delta PE = -\int F \, dx \quad , \quad \text{[one dimension]}
\]

so for electric fields we divide by \( q \) to find

\[
\Delta V = -\int E \, dx \quad , \quad \text{[one dimension]}
\]

Applying the fundamental theorem of calculus yields

\[
E = -\frac{dV}{dx} \quad . \quad \text{[one dimension]}
\]
What is the voltage associated with a point charge?

As derived previously in self-check A on page 131, the field is

\[ |E| = \frac{kQ}{r^2} \]

The difference in voltage between two points on the same radius line is

\[ \Delta V = \int dV = - \int E_x dx \]

In the general discussion above, \( x \) was just a generic name for distance traveled along the line from one point to the other, so in this case \( x \) really means \( r \).

\[ \Delta V = - \int_{r_1}^{r_2} E_x dr \]
\[ = - \int_{r_1}^{r_2} \frac{kQ}{r^2} dr \]
\[ = \frac{kQ}{r} \bigg|_{r_1}^{r_2} \]
\[ = \frac{kQ}{r_2} - \frac{kQ}{r_1} \]

The standard convention is to use \( r_1 = \infty \) as a reference point, so that the voltage at any distance \( r \) from the charge is

\[ V = \frac{kQ}{r} \]

The interpretation is that if you bring a positive test charge closer to a positive charge, its electrical energy is increased; if it was released, it would spring away, releasing this as kinetic energy.

**self-check B**

Show that you can recover the expression for the field of a point charge by evaluating the derivative \( E_x = -dV/dx \).  

\[ \text{Answer, p. 205} \]
The constant-voltage curves surrounding a point charge. Near the charge, the curves are so closely spaced that they blend together on this drawing due to the finite width with which they were drawn. Some electric fields are shown as arrows.

5.5 Two or Three Dimensions

The topographical map shown in figure p suggests a good way to visualize the relationship between field and voltage in two dimensions. Each contour on the map is a line of constant height; some of these are labeled with their elevations in units of feet. Height is related to gravitational potential energy, so in a gravitational analogy, we can think of height as representing voltage. Where the contour lines are far apart, as in the town, the slope is gentle. Lines close together indicate a steep slope.

If we walk along a straight line, say straight east from the town, then height (voltage) is a function of the east-west coordinate $x$. Using the usual mathematical definition of the slope, and writing $V$ for the height in order to remind us of the electrical analogy, the slope along such a line is $\Delta V/\Delta x$. If the slope isn’t constant, we either need to use the slope of the $V - x$ graph, or use calculus and talk about the derivative $dV/dx$.

What if everything isn’t confined to a straight line? Water flows downhill. Notice how the streams on the map cut perpendicularly through the lines of constant height.

It is possible to map voltages in the same way, as shown in figure q. The electric field is strongest where the constant-voltage curves are closest together, and the electric field vectors always point perpendicular to the constant-voltage curves.
Figure s shows some examples of ways to visualize field and voltage patterns.

Mathematically, the calculus of section 5.4 generalizes to three dimensions as follows:

\[
E_x = -\frac{dV}{dx} \\
E_y = -\frac{dV}{dy} \\
E_z = -\frac{dV}{dz}
\]

**self-check C**

Imagine that the topographical map in figure r represents voltage rather than height. (a) Consider the stream the starts near the center of the map. Determine the positive and negative signs of \(\frac{dV}{dx}\) and \(\frac{dV}{dy}\), and relate these to the direction of the force that is pushing the current forward against the resistance of friction. (b) If you wanted to find a lot of electric charge on this map, where would you look?  

\[\text{Answer, p. 206}\]
5.6 $\int \star$ Electric Field of a Continuous Charge Distribution

Charge really comes in discrete chunks, but often it is mathematically convenient to treat a set of charges as if they were like a continuous fluid spread throughout a region of space. For example, a charged metal ball will have charge spread nearly uniformly all over its surface, and in for most purposes it will make sense to ignore the fact that this uniformity is broken at the atomic level. The electric field made by such a continuous charge distribution is the sum of the fields created by every part of it. If we let the “parts” become infinitesimally small, we have a sum of an infinite number of infinitesimal numbers, which is an integral. If it was a discrete sum, we would have a total electric field in the $x$ direction that was the sum of all the $x$ components of the individual fields, and similarly we’d have sums for the $y$ and $z$ components. In the continuous case, we have three integrals.

$\text{Field of a uniformly charged rod}$

A rod of length $L$ has charge $Q$ spread uniformly along it. Find the electric field at a point a distance $d$ from the center of the rod, along the rod’s axis.

This is a one-dimensional situation, so we really only need to do a single integral representing the total field along the axis. We imagine breaking the rod down into short pieces of length $dz$, each with charge $dq$. Since charge is uniformly spread along the rod, we have $dq = \lambda dz$, where $\lambda = Q/L$ (Greek lambda) is the charge per unit length, in units of coulombs per meter. Since the pieces are infinitesimally short, we can treat them as point charges and use the expression $kdq/r^2$ for their contributions to the field, where $r = d - z$ is the distance from the charge at $z$ to the point in which we are interested.

$$E_z = \int \frac{k dq}{r^2} = \int_{-L/2}^{+L/2} \frac{k \lambda dz}{(d - z)^2} = k\lambda \int_{-L/2}^{+L/2} \frac{dz}{(d - z)^2}$$

The integral can be looked up in a table, or reduced to an elementary form by substituting a new variable for $d - z$. The result is

$$E_z = k\lambda \left( \frac{1}{d - z} \right)_{-L/2}^{+L/2} = \frac{kQ}{L} \left( \frac{1}{d - L/2} - \frac{1}{d + L/2} \right)$$
For large values of $d$, this expression gets smaller for two reasons: (1) the denominators of the fractions become large, and (2) the two fractions become nearly the same, and tend to cancel out. This makes sense, since the field should get weaker as we get farther away from the charge. In fact, the field at large distances must approach $kQ/d^2$, since from a great distance, the rod looks like a point.

It's also interesting to note that the field becomes infinite at the ends of the rod, but is not infinite on the interior of the rod. Can you explain physically why this happens?
Summary

Selected Vocabulary

field . . . . . . . . a property of a point in space describing the forces that would be exerted on a particle if it was there
sink . . . . . . . . a point at which field vectors converge
source . . . . . . . a point from which field vectors diverge; often used more inclusively to refer to points of either convergence or divergence
electric field . . the force per unit charge exerted on a test charge at a given point in space
gravitational field the force per unit mass exerted on a test mass at a given point in space
electric dipole . . an object that has an imbalance between positive charge on one side and negative charge on the other; an object that will experience a torque in an electric field

Notation

g . . . . . . . . the gravitational field
E . . . . . . . . the electric field
D . . . . . . . . an electric dipole moment

Other Terminology and Notation

d, p, m . . . . . . other notations for the electric dipole moment

Summary

Newton conceived of a universe where forces reached across space instantaneously, but we now know that there is a delay in time before a change in the configuration of mass and charge in one corner of the universe will make itself felt as a change in the forces experienced far away. We imagine the outward spread of such a change as a ripple in an invisible universe-filling field of force.

We define the gravitational field at a given point as the force per unit mass exerted on objects inserted at that point, and likewise the electric field is defined as the force per unit charge. These fields are vectors, and the fields generated by multiple sources add according to the rules of vector addition.

When the electric field is constant, the voltage difference between two points lying on a line parallel to the field is related to the field by the equation $\Delta V = -Ed$, where $d$ is the distance between the two points.
1. In our by-now-familiar neuron, the voltage difference between the inner and outer surfaces of the cell membrane is about $V_{\text{out}} - V_{\text{in}} = -70 \, \text{mV}$ in the resting state, and the thickness of the membrane is about $6.0 \, \text{nm}$ (i.e., only about a hundred atoms thick). What is the electric field inside the membrane? √

2. The gap between the electrodes in an automobile engine’s spark plug is $0.060 \, \text{cm}$. To produce an electric spark in a gasoline-air mixture, an electric field of $3.0 \times 10^6 \, \text{V/m}$ must be achieved. On starting a car, what minimum voltage must be supplied by the ignition circuit? Assume the field is uniform. √

(b) The small size of the gap between the electrodes is inconvenient because it can get blocked easily, and special tools are needed to measure it. Why don’t they design spark plugs with a wider gap?

3. (a) At time $t = 0$, a positively charged particle is placed, at rest, in a vacuum, in which there is a uniform electric field of magnitude $E$. Write an equation giving the particle’s speed, $v$, in terms of $t$, $E$, and its mass and charge $m$ and $q$. √

(b) If this is done with two different objects and they are observed to have the same motion, what can you conclude about their masses and charges? (For instance, when radioactivity was discovered, it was found that one form of it had the same motion as an electron in this type of experiment.)

4. Show that the magnitude of the electric field produced by a simple two-charge dipole, at a distant point along the dipole’s axis, is to a good approximation proportional to $D/r^3$, where $r$ is the distance from the dipole. [Hint: Use the approximation $(1 + \epsilon)^p \approx 1 + p\epsilon$, which is valid for small $\epsilon$.] ⋆

5. Given that the field of a dipole is proportional to $D/r^3$ (see previous problem), show that its voltage varies as $D/r^2$. (Ignore positive and negative signs and numerical constants of proportionality.) ∫

6. A carbon dioxide molecule is structured like O-C-O, with all three atoms along a line. The oxygen atoms grab a little bit of extra negative charge, leaving the carbon positive. The molecule’s symmetry, however, means that it has no overall dipole moment, unlike a V-shaped water molecule, for instance. Whereas the voltage of a dipole of magnitude $D$ is proportional to $D/r^2$ (problem 5), it turns out that the voltage of a carbon dioxide molecule along its axis equals $k/r^3$, where $r$ is the distance from the molecule and $k$
is a constant. What would be the electric field of a carbon dioxide molecule at a distance $r$?

7 A proton is in a region in which the electric field is given by $E = a + bx^3$. If the proton starts at rest at $x_1 = 0$, find its speed, $v$, when it reaches position $x_2$. Give your answer in terms of $a$, $b$, $x_2$, and $e$ and $m$, the charge and mass of the proton.

8 Consider the electric field created by a uniform ring of total charge $q$ and radius $b$. (a) Show that the field at a point on the ring’s axis at a distance $a$ from the plane of the ring is $kqa(a^2 + b^2)^{-3/2}$. (b) Show that this expression has the right behavior for $a = 0$ and for $a$ much greater than $b$.

9 Consider the electric field created by an infinite uniformly charged plane. Starting from the result of problem 8, show that the field at any point is $2\pi k\sigma$, where $\sigma$ is the density of charge on the plane, in units of coulombs per square meter. Note that the result is independent of the distance from the plane. [Hint: Slice the plane into infinitesimal concentric rings, centered at the point in the plane closest to the point at which the field is being evaluated. Integrate the rings’ contributions to the field at this point to find the total field.]

10 Consider the electric field created by a uniformly charged cylinder that extends to infinity in one direction. (a) Starting from the result of problem 8, show that the field at the center of the cylinder’s mouth is $2\pi k\sigma$, where $\sigma$ is the density of charge on the cylinder, in units of coulombs per square meter. [Hint: You can use a method similar to the one in problem 9.] (b) This expression is independent of the radius of the cylinder. Explain why this should be so. For example, what would happen if you doubled the cylinder’s radius?

11 Three charges are arranged on a square as shown. All three charges are positive. What value of $q_2/q_1$ will produce zero electric field at the center of the square?

▷ Solution, p. 207

Problem 11.
Chapter 6

Electromagnetism

In this chapter we discuss the intimate relationship between magnetism and electricity discovered by James Clerk Maxwell. Maxwell
realized that light was a wave made up of electric and magnetic fields linked to each other. He is said to have gone for a walk with his wife one night and told her that she was the only other person in the world who knew what starlight really was.

### 6.1 The Magnetic Field

**No magnetic monopoles**

If you could play with a handful of electric dipoles and a handful of bar magnets, they would appear very similar. For instance, a pair of bar magnets wants to align themselves head-to-tail, and a pair of electric dipoles does the same thing. (It is unfortunately not that easy to make a permanent electric dipole that can be handled like this, since the charge tends to leak.)

You would eventually notice an important difference between the two types of objects, however. The electric dipoles can be broken apart to form isolated positive charges and negative charges. The two-ended device can be broken into parts that are not two-ended. But if you break a bar magnet in half, b, you will find that you have simply made two smaller two-ended objects.

The reason for this behavior is not hard to divine from our microscopic picture of permanent iron magnets. An electric dipole has extra positive “stuff” concentrated in one end and extra negative in the other. The bar magnet, on the other hand, gets its magnetic properties not from an imbalance of magnetic “stuff” at the two ends but from the orientation of the rotation of its electrons. One end is the one from which we could look down the axis and see the electrons rotating clockwise, and the other is the one from which they would appear to go counterclockwise. There is no difference between the “stuff” in one end of the magnet and the other, c.

Nobody has ever succeeded in isolating a single magnetic pole. In technical language, we say that magnetic monopoles do not seem to exist. Electric monopoles do exist — that’s what charges are.

Electric and magnetic forces seem similar in many ways. Both act at a distance, both can be either attractive or repulsive, and both are intimately related to the property of matter called charge. (Recall that magnetism is an interaction between moving charges.) Physicists’s aesthetic senses have been offended for a long time because this seeming symmetry is broken by the existence of electric monopoles and the absence of magnetic ones. Perhaps some exotic form of matter exists, composed of particles that are magnetic monopoles. If such particles could be found in cosmic rays or moon rocks, it would be evidence that the apparent asymmetry was only an asymmetry in the composition of the universe, not in the laws of physics. For these admittedly subjective reasons, there have been several searches for magnetic monopoles.
have been performed, with negative results, to look for magnetic monopoles embedded in ordinary matter. Soviet physicists in the 1960s made exciting claims that they had created and detected magnetic monopoles in particle accelerators, but there was no success in attempts to reproduce the results there or at other accelerators. The most recent search for magnetic monopoles, done by reanalyzing data from the search for the top quark at Fermilab, turned up no candidates, which shows that either monopoles don’t exist in nature or they are extremely massive and thus hard to create in accelerators.

**Definition of the magnetic field**

Since magnetic monopoles don’t seem to exist, it would not make much sense to define a magnetic field in terms of the force on a test monopole. Instead, we follow the philosophy of the alternative definition of the electric field, and define the field in terms of the torque on a magnetic test dipole. This is exactly what a magnetic compass does: the needle is a little iron magnet which acts like a magnetic dipole and shows us the direction of the earth’s magnetic field.

To define the strength of a magnetic field, however, we need some way of defining the strength of a test dipole, i.e., we need a definition of the magnetic dipole moment. We could use an iron permanent magnet constructed according to certain specifications, but such an object is really an extremely complex system consisting of many iron atoms, only some of which are aligned. A more fundamental standard dipole is a square current loop. This could be little resistive circuit consisting of a square of wire shorting across a battery.

We will find that such a loop, when placed in a magnetic field, experiences a torque that tends to align plane so that its face points in a certain direction. (Since the loop is symmetric, it doesn’t care if we rotate it like a wheel without changing the plane in which it lies.) It is this preferred facing direction that we will end up defining as the direction of the magnetic field.

Experiments show if the loop is out of alignment with the field, the torque on it is proportional to the amount of current, and also to the interior area of the loop. The proportionality to current makes sense, since magnetic forces are interactions between moving charges, and current is a measure of the motion of charge. The proportionality to the loop’s area is also not hard to understand, because increasing the length of the sides of the square increases both the amount of charge contained in this circular “river” and the amount of leverage supplied for making torque. Two separate physical reasons for a proportionality to length result in an overall proportionality to length squared, which is the same as the area of the loop. For these reasons, we define the magnetic dipole moment
The unit of magnetic field, the tesla, is named after Serbian-American inventor Nikola Tesla.

The magnetic field pattern of a bar magnet. This picture was made by putting iron filings on a piece of paper, and bringing a bar magnet up underneath it. Note how the field pattern passes across the body of the magnet, forming closed loops, as in figure h/2. There are no sources or sinks.

We now define the magnetic field in a manner entirely analogous to the second definition of the electric field:

**definition of the magnetic field**

The magnetic field vector, \( \mathbf{B} \), at any location in space is defined by observing the torque exerted on a magnetic test dipole \( D_m \) consisting of a square current loop. The field’s magnitude is \( |\mathbf{B}| = \frac{\tau}{D_m} \sin \theta \), where \( \theta \) is the angle by which the loop is misaligned. The direction of the field is perpendicular to the loop; of the two perpendiculars, we choose the one such that if we look along it, the loop’s current is counterclockwise.

We find from this definition that the magnetic field has units of \( \text{N} \cdot \text{m} / \text{A} \cdot \text{m}^2 = \text{N/A} \cdot \text{m} \). This unwieldy combination of units is abbreviated as the tesla, \( 1 \text{T} = 1 \text{N/A} \cdot \text{m} \). Refrain from memorizing the part about the counterclockwise direction at the end; in section 6.4 we’ll see how to understand this in terms of more basic principles.

The nonexistence of magnetic monopoles means that unlike an electric field, h/1, a magnetic one, h/2, can never have sources or sinks. The magnetic field vectors lead in paths that loop back on themselves, without ever converging or diverging at a point.

### 6.2 Calculating Magnetic Fields and Forces

**Magnetostatics**

Our study of the electric field built on our previous understanding of electric forces, which was ultimately based on Coulomb’s law for the electric force between two point charges. Since magnetism is ultimately an interaction between currents, i.e., between moving charges, it is reasonable to wish for a magnetic analog of Coulomb’s law, an equation that would tell us the magnetic force between any two moving point charges.

Such a law, unfortunately, does not exist. Coulomb’s law describes the special case of electrostatics: if a set of charges is sitting around and not moving, it tells us the interactions among them. Coulomb’s law fails if the charges are in motion, since it does not incorporate any allowance for the time delay in the outward propagation of a change in the locations of the charges.

A pair of moving point charges will certainly exert magnetic forces on one another, but their magnetic fields are like the v-shaped bow waves left by boats. Each point charge experiences a magnetic field that originated from the other charge when it was at some previous position. There is no way to construct a force law that tells
us the force between them based only on their current positions in space.

There is, however, a science of magnetostatics that covers a great many important cases. Magnetostatics describes magnetic forces among currents in the special case where the currents are steady and continuous, leading to magnetic fields throughout space that do not change over time.

If we cannot build a magnetostatics from a force law for point charges, then where do we start? It can be done, but the level of mathematics required (vector calculus) is inappropriate for this course. Luckily there is an alternative that is more within our reach. Physicists of generations past have used the fancy math to derive simple equations for the fields created by various static current distributions, such as a coil of wire, a circular loop, or a straight wire. Virtually all practical situations can be treated either directly using these equations or by doing vector addition, e.g., for a case like the field of two circular loops whose fields add onto one another. Figure i shows the equations for some of the more commonly encountered configurations, with illustrations of their field patterns.

Field created by a long, straight wire carrying current I:

\[ B = \frac{\mu_0 I}{2\pi r} \]

Here \( r \) is the distance from the center of the wire. The field vectors trace circles in planes perpendicular to the wire, going clockwise when viewed from along the direction of the current.

Field created by a single circular loop of current:

The field vectors form a dipole-like pattern, coming through the loop and back around on the outside. Each oval path traced out by the field vectors appears clockwise if viewed from along the direction the current is going when it punches through it. There is no simple equation for a field at an arbitrary point in space, but for a point lying along the central axis perpendicular to the loop, the field is

\[ B = \frac{1}{2} \mu_0 I b^2 \left( b^2 + z^2 \right)^{-3/2} \]

where \( b \) is the radius of the loop and \( z \) is the distance of the point from the plane of the loop.

Field created by a solenoid (cylindrical coil):

The field pattern is similar to that of a single loop, but for a long solenoid the paths of the field vectors become very straight on the inside of the coil and on the outside immediately next to the coil. For a sufficiently long solenoild, the interior field also becomes very nearly uniform, with a magnitude of

\[ B = \frac{\mu_0 I N}{\ell} \]

where \( N \) is the number of turns of wire and \( \ell \) is the length of the solenoid. The field near the mouths or outside the coil is not constant, and is more difficult to calculate. For a long solenoid, the exterior field is much smaller than the interior field.

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Don’t memorize the equations! The symbol $\mu_o$ is an abbreviation for the constant $4\pi \times 10^{-7}$ T·m/A. It is the magnetic counterpart of the Coulomb force constant $k$. The Coulomb constant tells us how much electric field is produced by a given amount of charge, while $\mu_o$ relates currents to magnetic fields. Unlike $k$, $\mu_o$ has a definite numerical value because of the design of the metric system.

**Force on a charge moving through a magnetic field**

We now know how to calculate magnetic fields in some typical situations, but one might also like to be able to calculate magnetic forces, such as the force of a solenoid on a moving charged particle, or the force between two parallel current-carrying wires.

We will restrict ourselves to the case of the force on a charged particle moving through a magnetic field, which allows us to calculate the force between two objects when one is a moving charged particle and the other is one whose magnetic field we know how to find. An example is the use of solenoids inside a TV tube to guide the electron beam as it paints a picture.

Experiments show that the magnetic force on a moving charged particle has a magnitude given by

$$|F| = q|v||B|\sin \theta,$$

where $v$ is the velocity vector of the particle, and $\theta$ is the angle between the $v$ and $B$ vectors. Unlike electric and gravitational forces, magnetic forces do not lie along the same line as the field vector. The force is always perpendicular to both $v$ and $B$. Given two vectors, there is only one line perpendicular to both of them, so the force vector points in one of the two possible directions along this line. For a positively charged particle, the direction of the force vector can be found as follows. First, position the $v$ and $B$ vectors with their tails together. The direction of $F$ is such that if you sight along it, the $B$ vector is clockwise from the $v$ vector; for a negatively charged particle the direction of the force is reversed. Note that since the force is perpendicular to the particle’s motion, the magnetic field never does work on it.

**Magnetic levitation example 1**

In figure 1, a small, disk-shaped permanent magnet is stuck on the side of a battery, and a wire is clasped loosely around the battery, shorting it. A large current flows through the wire. The electrons moving through the wire feel a force from the magnetic field made by the permanent magnet, and this force levitates the wire.

From the photo, it’s possible to find the direction of the magnetic field made by the permanent magnet. The electrons in the copper wire are negatively charged, so they flow from the negative (flat) terminal of the battery to the positive terminal (the one with the bump, in front). As the electrons pass by the permanent magnet,
we can imagine that they would experience a field either toward the magnet, or away from it, depending on which way the magnet was flipped when it was stuck onto the battery. Imagine sighting along the upward force vector, which you could do if you were a tiny bug lying on your back underneath the wire. Since the electrons are negatively charged, the $B$ vector must be counterclockwise from the $v$ vector, which means toward the magnet.

1A circular orbit example 2
Magnetic forces cause a beam of electrons to move in a circle. The beam is created in a vacuum tube, in which a small amount of hydrogen gas has been left. A few of the electrons strike hydrogen molecules, creating light and letting us see the beam. A magnetic field is produced by passing a current (meter) through the circular coils of wire in front of and behind the tube. In the bottom figure, with the magnetic field turned on, the force perpendicular to the electrons’ direction of motion causes them to move in a circle.

1Hallucinations during an MRI scan example 3
During an MRI scan of the head, the patient’s nervous system is exposed to intense magnetic fields. The average velocities of the charge-carrying ions in the nerve cells is fairly low, but if the patient moves her head suddenly, the velocity can be high enough that the magnetic field makes significant forces on the ions. This can result in visual and auditory hallucinations, e.g., frying bacon sounds.

6.3 Induction
Electromagnetism and relative motion

The theory of electric and magnetic fields constructed up to this point contains a paradox. One of the most basic principles of physics, dating back to Newton and Galileo and still going strong today, states that motion is relative, not absolute. Thus the laws of physics should not function any differently in a moving frame of reference, or else we would be able to tell which frame of reference was the one in an absolute state of rest. As an example from mechanics, imagine that a child is tossing a ball up and down in the back seat of a moving car. In the child’s frame of reference, the car is at rest and the landscape is moving by; in this frame, the ball goes straight up and down, and obeys Newton’s laws of motion and Newton’s law of gravity. In the frame of reference of an observer watching from the sidewalk, the car is moving and the sidewalk isn’t. In this frame, the ball follows a parabolic arc, but it still obeys Newton’s laws.

When it comes to electricity and magnetism, however, we have a problem, which was first clearly articulated by Einstein: if we state that magnetism is an interaction between moving charges, we have
Observer A sees a positively charged particle moves through a region of upward magnetic field, which we assume to be uniform, between the poles of two magnets. The resulting force along the $z$ axis causes the particle’s path to curve toward us.

apparently created a law of physics that violates the principle that motion is relative, since different observers in different frames would disagree about how fast the charges were moving, or even whether they were moving at all. The incorrect solution that Einstein was taught (and disbelieved) as a student around the year 1900 was that the relative nature of motion applied only to mechanics, not to electricity and magnetism. The full story of how Einstein restored the principle of relative motion to its rightful place in physics involves his theory of special relativity, which we will not take up until book 6 of this series. However, a few simple and qualitative thought experiments will suffice to show how, based on the principle that motion is relative, there must be some new and previously unsuspected relationships between electricity and magnetism. These relationships form the basis for many practical, everyday devices, such as generators and transformers, and they also lead to an explanation of light itself as an electromagnetic phenomenon.

Let’s imagine an electrical example of relative motion in the same spirit as the story of the child in the back of the car. Suppose we have a line of positive charges, $m$. Observer A is in a frame of reference which is at rest with respect to these charges, and observes that they create an electric field pattern that points outward, away from the charges, in all directions, like a bottle brush. Suppose, however, that observer B is moving to the right with respect to the charges. As far as B is concerned, she’s the one at rest, while the charges (and observer A) move to the left. In agreement with A, she observes an electric field, but since to her the charges are in motion, she must also observe a magnetic field in the same region of space, exactly like the magnetic field made by a current in a long, straight wire.

Who’s right? They’re both right. In A’s frame of reference, there is only an $E$, while in B’s frame there is both an $E$ and a $B$. The principle of relative motion forces us to conclude that depending on our frame of reference we will observe a different combination of fields. Although we will not prove it (the proof requires special relativity, which we get to in book 6), it is true that either frame of reference provides a perfectly self-consistent description of things. For instance, if an electron passes through this region of space, both A and B will see it swerve, speed up, and slow down. A will successfully explain this as the result of an electric field, while B will ascribe the electron’s behavior to a combination of electric and magnetic forces.

Thus, if we believe in the principle of relative motion, then we must accept that electric and magnetic fields are closely related phenomena, two sides of the same coin.

Now consider figure n. Observer A is at rest with respect to the bar magnets, and sees the particle swerving off in the $z$ direction, as it should according to the rule given in section 6.2 (sighting along
the force vector, i.e., from behind the page, the \( B \) vector is clockwise from the \( v \) vector. Suppose observer B, on the other hand, is moving to the right along the \( x \) axis, initially at the same speed as the particle. B sees the bar magnets moving to the left and the particle initially at rest but then accelerating along the \( z \) axis in a straight line. It is not possible for a magnetic field to start a particle moving if it is initially at rest, since magnetism is an interaction of moving charges with moving charges. B is thus led to the inescapable conclusion that there is an electric field in this region of space, which points along the \( z \) axis. In other words, what A perceives as a pure \( B \) field, B sees as a mixture of \( E \) and \( B \).

In general, observers who are not at rest with respect to one another will perceive different mixtures of electric and magnetic fields.

**The principle of induction**

So far everything we’ve been doing might not seem terribly useful, since it seems that nothing surprising will happen as long as we stick to a single frame of reference, and don’t worry about what people in other frames think. That isn’t the whole story, however, as was discovered experimentally by Faraday in 1831 and explored mathematically by Maxwell later in the same century. Let’s state Faraday’s idea first, and then see how something like it must follow inevitably from the principle that motion is relative:

**the principle of induction**

Any electric field that changes over time will produce a magnetic field in the space around it.

Any magnetic field that changes over time will produce an electric field in the space around it.

The induced field tends to have a whirlpool pattern, as shown in figure o, but the whirlpool image is not to be taken too literally; the principle of induction really just requires a field pattern such that, if one inserted a paddlewheel in it, the paddlewheel would spin. All of the field patterns shown in figure p are ones that could be created by induction; all have a counterclockwise “curl” to them.
Observer A is at rest with respect to the bar magnet, and observes magnetic fields that have different strengths at different distances from the magnet. Observer B, hanging out in the region to the left of the magnet, sees the magnet moving toward her, and detects that the magnetic field in that region is getting stronger as time passes. As in 1, there is an electric field along the z axis because she’s in motion with respect to the magnet. The $\Delta B$ vector is upward, and the electric field has a curliness to it: a paddlewheel inserted in the electric field would spin clockwise as seen from above, since the clockwise torque made by the strong electric field on the right is greater than the counterclockwise torque made by the weaker electric field on the left.

Figure q shows an example of the fundamental reason why a changing $B$ field must create an $E$ field. The electric field would be inexplicable to observer B if she believed only in Coulomb’s law, and thought that all electric fields are made by electric charges. If she knows about the principle of induction, however, the existence of this field is to be expected.

A generator, $r$, consists of a permanent magnet that rotates within a coil of wire. The magnet is turned by a motor or crank, (not shown). As it spins, the nearby magnetic field changes. According to the principle of induction, this changing magnetic field results in an electric field, which has a whirlpool pattern. This electric field pattern creates a current that whips around the coils of wire, and we can tap this current to light the lightbulb.

When you’re driving a car, the engine recharges the battery continuously using a device called an alternator, which is really just a generator like the one shown on the previous page, except that the coil rotates while the permanent magnet is fixed in place. Why can’t you use the alternator to start the engine if your car’s battery is dead?  

The transformer works with alternating current, so the magnetic field surrounding the input coil is always changing. This induces an electric field, which drives a current around the output coil.

If both coils were the same, the arrangement would be symmetric, and the output would be the same as the input, but an output coil with a smaller number of coils gives the electric forces a smaller distance through which to push the electrons. Less mechanical work per unit charge means a lower voltage. Conservation of en-
ergy, however, guarantees that the amount of power on the output side must equal the amount put in originally, $i_{in}V_{in} = i_{out}V_{out}$, so this reduced voltage must be accompanied by an increased current.

A mechanical analogy example 6
Figure 5 shows an example of induction (left) with a mechanical analogy (right). The two bar magnets are initially pointing in opposite directions, 1, and their magnetic fields cancel out. If one magnet is flipped, 2, their fields reinforce, but the change in the magnetic field takes time to spread through space. Eventually, 3, the field becomes what you would expect from the theory of magnetostatics. In the mechanical analogy, the sudden motion of the hand produces a violent kink or wave pulse in the rope, the pulse travels along the rope, and it takes some time for the rope to settle down. An electric field is also induced in by the changing magnetic field, even though there is no net charge anywhere to act as a source. (These simplified drawings are not meant to be accurate representations of the complete three-dimensional pattern of electric and magnetic fields.)

Discussion Question
A In figures n and q, observer B is moving to the right. What would have happened if she had been moving to the left?
6.4 Electromagnetic Waves

The most important consequence of induction is the existence of electromagnetic waves. Whereas a gravitational wave would consist of nothing more than a rippling of gravitational fields, the principle of induction tells us that there can be no purely electrical or purely magnetic waves. Instead, we have waves in which there are both electric and magnetic fields, such as the sinusoidal one shown in the figure. Maxwell proved that such waves were a direct consequence of his equations, and derived their properties mathematically. The derivation would be beyond the mathematical level of this book, so we will just state the results.

A sinusoidal electromagnetic wave has the geometry shown in figure t. The E and B fields are perpendicular to the direction of motion, and are also perpendicular to each other. If you look along the direction of motion of the wave, the B vector is always 90 degrees clockwise from the E vector. The magnitudes of the two fields are related by the equation $|E| = c|B|$.

How is an electromagnetic wave created? It could be emitted, for example, by an electron orbiting an atom or currents going back and forth in a transmitting antenna. In general any accelerating charge will create an electromagnetic wave, although only a current that varies sinusoidally with time will create a sinusoidal wave. Once created, the wave spreads out through space without any need for charges or currents along the way to keep it going. As the electric field oscillates back and forth, it induces the magnetic field, and the oscillating magnetic field in turn creates the electric field. The whole wave pattern propagates through empty space at a velocity $c = 3.0 \times 10^8$ m/s, which is related to the constants $k$ and $\mu_0$ by $c = \sqrt{\frac{4\pi k}{\mu_0}}$.

Polarization

Two electromagnetic waves traveling in the same direction through space can differ by having their electric and magnetic fields in different directions, a property of the wave called its polarization.
Light is an electromagnetic wave

Once Maxwell had derived the existence of electromagnetic waves, he became certain that they were the same phenomenon as light. Both are transverse waves (i.e., the vibration is perpendicular to the direction the wave is moving), and the velocity is the same.

Heinrich Hertz (for whom the unit of frequency is named) verified Maxwell’s ideas experimentally. Hertz was the first to succeed in producing, detecting, and studying electromagnetic waves in detail using antennas and electric circuits. To produce the waves, he had to make electric currents oscillate very rapidly in a circuit. In fact, there was really no hope of making the current reverse directions at the frequencies of $10^{15}$ Hz possessed by visible light. The fastest electrical oscillations he could produce were $10^9$ Hz, which would give a wavelength of about 30 cm. He succeeded in showing that, just like light, the waves he produced were polarizable, and could be reflected and refracted (i.e., bent, as by a lens), and he built devices such as parabolic mirrors that worked according to the same optical principles as those employing light. Hertz’s results were convincing evidence that light and electromagnetic waves were one and the same.

The electromagnetic spectrum

Today, electromagnetic waves with frequencies in the range employed by Hertz are known as radio waves. Any remaining doubts that the “Hertzian waves,” as they were then called, were the same type of wave as light waves were soon dispelled by experiments in the whole range of frequencies in between, as well as the frequencies outside that range. In analogy to the spectrum of visible light, we speak of the entire electromagnetic spectrum, of which the visible spectrum is one segment.

The terminology for the various parts of the spectrum is worth memorizing, and is most easily learned by recognizing the logical re-
relationships between the wavelengths and the properties of the waves with which you are already familiar. Radio waves have wavelengths that are comparable to the size of a radio antenna, i.e., meters to tens of meters. Microwaves were named that because they have much shorter wavelengths than radio waves; when food heats unevenly in a microwave oven, the small distances between neighboring hot and cold spots is half of one wavelength of the standing wave the oven creates. The infrared, visible, and ultraviolet obviously have much shorter wavelengths, because otherwise the wave nature of light would have been as obvious to humans as the wave nature of ocean waves. To remember that ultraviolet, x-rays, and gamma rays all lie on the short-wavelength side of visible, recall that all three of these can cause cancer. (As we’ll discuss later in the course, there is a basic physical reason why the cancer-causing disruption of DNA can only be caused by very short-wavelength electromagnetic waves. Contrary to popular belief, microwaves cannot cause cancer, which is why we have microwave ovens and not x-ray ovens!)

\[ Why \, the \, sky \, is \, blue \, example \, 7 \]  
When sunlight enters the upper atmosphere, a particular air molecule finds itself being washed over by an electromagnetic wave of frequency \( f \). The molecule’s charged particles (nuclei and electrons) act like oscillators being driven by an oscillating force, and respond by vibrating at the same frequency \( f \). Energy is sucked out of the incoming beam of sunlight and converted into the kinetic energy of the oscillating particles. However, these particles are accelerating, so they act like little radio antennas that put the energy back out as spherical waves of light that spread out in all directions. An object oscillating at a frequency \( f \) has an acceleration proportional to \( f^2 \), and an accelerating charged particle creates an electromagnetic wave whose fields are proportional to its acceleration, so the field of the reradiated spherical wave is proportional to \( f^2 \). The energy of a field is proportional to the square of the field, so the energy of the reradiated is proportional to \( f^4 \). Since blue light has about twice the frequency of red light, this process is about \( 2^4 = 16 \) times as strong for blue as for red, and that’s why the sky is blue.

6.5 Calculating Energy In Fields  
We have seen that the energy stored in a wave (actually the energy density) is typically proportional to the square of the wave’s amplitude. Fields of force can make wave patterns, for which we might expect the same to be true. This turns out to be true not only for
wave-like field patterns but for all fields:

\[
\begin{align*}
\text{energy stored in the gravitational field per } m^3 &= -\frac{1}{8\pi G}|g|^2 \\
\text{energy stored in the electric field per } m^3 &= \frac{1}{8\pi k}|E|^2 \\
\text{energy stored in the magnetic field per } m^3 &= \frac{1}{2\mu_0}|B|^2
\end{align*}
\]

Although funny factors of \(8\pi\) and the plus and minus signs may have initially caught your eye, they are not the main point. The important idea is that the energy density is proportional to the square of the field strength in all three cases. We first give a simple numerical example and work a little on the concepts, and then turn our attention to the factors out in front.

\textit{Getting killed by a solenoid} example 8

Solenoids are very common electrical devices, but they can be a hazard to someone who is working on them. Imagine a solenoid that initially has a DC current passing through it. The current creates a magnetic field inside and around it, which contains energy. Now suppose that we break the circuit. Since there is no longer a complete circuit, current will quickly stop flowing, and the magnetic field will collapse very quickly. The field had energy stored in it, and even a small amount of energy can create a dangerous power surge if released over a short enough time interval. It is prudent not to fiddle with a solenoid that has current flowing through it, since breaking the circuit could be hazardous to your health.

As a typical numerical estimate, let’s assume a 40 cm \(\times\) 40 cm \(\times\) 40 cm solenoid with an interior magnetic field of 1.0 T (quite a strong field). For the sake of this rough estimate, we ignore the exterior field, which is weak, and assume that the solenoid is cubical in shape. The energy stored in the field is

\[
\text{(energy per unit volume)(volume)} = \frac{1}{2\mu_0}|B|^2 V
\]

\[
= 3 \times 10^4 \text{ J}
\]

That’s a lot of energy!

In chapter 5 when we discussed the original reason for introducing the concept of a field of force, a prime motivation was that otherwise there was no way to account for the energy transfers involved when forces were delayed by an intervening distance. We used to think of the universe’s energy as consisting of
kinetic energy
+gravitational potential energy based on the distances between objects that interact gravitationally
+electric potential energy based on the distances between objects that interact electrically
+magnetic potential energy based on the distances between objects that interact magnetically

but in nonstatic situations we must use a different method:

kinetic energy
+gravitational potential energy stored in gravitational fields
+electric potential energy stored in electric fields
+magnetic potential stored in magnetic fields
Surprisingly, the new method still gives the same answers for the static cases.

**Energy stored in a capacitor**

Example 9

A pair of parallel metal plates, seen from the side in figure v, can be used to store electrical energy by putting positive charge on one side and negative charge on the other. Such a device is called a capacitor. (We have encountered such an arrangement previously, but there its purpose was to deflect a beam of electrons, not to store energy.)

In the old method of describing potential energy, 1, we think in terms of the mechanical work that had to be done to separate the positive and negative charges onto the two plates, working against their electrical attraction. The new description, 2, attributes the storage of energy to the newly created electric field occupying the volume between the plates. Since this is a static case, both methods give the same, correct answer.

**Potential energy of a pair of opposite charges**

Example 10

Imagine taking two opposite charges, w, that were initially far apart and allowing them to come together under the influence of their electrical attraction.

According to the old method, potential energy is lost because the electric force did positive work as it brought the charges together. (This makes sense because as they come together and accelerate it is their potential energy that is being lost and converted to kinetic energy.)

By the new method, we must ask how the energy stored in the electric field has changed. In the region indicated approximately by the shading in the figure, the superposing fields of the two charges undergo partial cancellation because they are in opposing directions. The energy in the shaded region is reduced by this effect. In the unshaded region, the fields reinforce, and the energy is increased.

It would be quite a project to do an actual numerical calculation of the energy gained and lost in the two regions (this is a case where the old method of finding energy gives greater ease of computation), but it is fairly easy to convince oneself that the energy is less when the charges are closer. This is because bringing the charges together shrinks the high-energy unshaded region and enlarges the low-energy shaded region.

**Energy in an electromagnetic wave**

Example 11

The old method would give zero energy for a region of space containing an electromagnetic wave but no charges. That would be wrong! We can only use the old method in static cases.

Now let’s give at least some justification for the other features of the three expressions for energy density, $-\frac{1}{8\pi\varepsilon_0}|\mathbf{E}|^2$, $\frac{1}{8\pi\varepsilon_0}|\mathbf{E}|^2$, and...
\( -\frac{1}{\mu_0} |B|^2 \), besides the proportionality to the square of the field strength.

First, why the different plus and minus signs? The basic idea is that the signs have to be opposite in the gravitational and electric cases because there is an attraction between two positive masses (which are the only kind that exist), but two positive charges would repel. Since we’ve already seen examples where the positive sign in the electric energy makes sense, the gravitational energy equation must be the one with the minus sign.

It may also seem strange that the constants \( G, k, \) and \( \mu_0 \) are in the denominator. They tell us how strong the three different forces are, so shouldn’t they be on top? No. Consider, for instance, an alternative universe in which gravity is twice as strong as in ours. The numerical value of \( G \) is doubled. Because \( G \) is doubled, all the gravitational field strengths are doubled as well, which quadruples the quantity \(|g|^2\). In the expression \(-\frac{1}{8\pi G} |g|^2\), we have quadrupled something on top and doubled something on the bottom, which makes the energy twice as big. That makes perfect sense.

**Discussion Questions**

**A.** The figure shows a positive charge in the gap between two capacitor plates. First make a large drawing of the field pattern that would be formed by the capacitor itself, without the extra charge in the middle. Next, show how the field pattern changes when you add the particle at these two positions. Compare the energy of the electric fields in the two cases. Does this agree with what you would have expected based on your knowledge of electrical forces?

**B.** Criticize the following statement: “A solenoid makes a charge in the space surrounding it, which dissipates when you release the energy.”

**C.** In example 10, I argued that the fields surrounding a positive and negative charge contain less energy when the charges are closer together. Perhaps a simpler approach is to consider the two extreme possibilities: the case where the charges are infinitely far apart, and the one in which they are at zero distance from each other, i.e., right on top of each other. Carry out this reasoning for the case of (1) a positive charge and a negative charge of equal magnitude, (2) two positive charges of equal magnitude, (3) the gravitational energy of two equal masses.

### 6.6  Symmetry and Handedness

The physicist Richard Feynman helped to get me hooked on physics with an educational film containing the following puzzle. Imagine that you establish radio contact with an alien on another planet. Neither of you even knows where the other one’s planet is, and you aren’t able to establish any landmarks that you both recognize. You manage to learn quite a bit of each other’s languages, but you’re stumped when you try to establish the definitions of left and right (or, equivalently, clockwise and counterclockwise). Is there any way to do it?
If there was any way to do it without reference to external landmarks, then it would imply that the laws of physics themselves were asymmetric, which would be strange. Why should they distinguish left from right? The gravitational field pattern surrounding a star or planet looks the same in a mirror, and the same goes for electric fields. However, the field patterns shown in section 6.2 seem to violate this principle, but do they really? Could you use these patterns to explain left and right to the alien? In fact, the answer is no. If you look back at the definition of the magnetic field in section 6.1, it also contains a reference to handedness: the counterclockwise direction of the loop’s current as viewed along the magnetic field. The aliens might have reversed their definition of the magnetic field, in which case their drawings of field patterns would look like mirror images of ours.

Until the middle of the twentieth century, physicists assumed that any reasonable set of physical laws would have to have this kind of symmetry between left and right. An asymmetry would be grotesque. Whatever their aesthetic feelings, they had to change their opinions about reality when experiments showed that the weak nuclear force (section 6.5) violates right-left symmetry! It is still a mystery why right-left symmetry is observed so scrupulously in general, but is violated by one particular type of physical process.
Summary

Selected Vocabulary

magnetic field . . . a field of force, defined in terms of the torque exerted on a test dipole
magnetic dipole . an object, such as a current loop, an atom, or a bar magnet, that experiences torques due to magnetic forces; the strength of magnetic dipoles is measured by comparison with a standard dipole consisting of a square loop of wire of a given size and carrying a given amount of current
induction . . . . . . the production of an electric field by a changing magnetic field, or vice-versa

Notation

$B$ . . . . . . . . . the magnetic field
$D_m$ . . . . . . . . . magnetic dipole moment

Summary

Magnetism is an interaction of moving charges with other moving charges. The magnetic field is defined in terms of the torque on a magnetic test dipole. It has no sources or sinks; magnetic field patterns never converge on or diverge from a point.

The magnetic and electric fields are intimately related. The principle of induction states that any changing electric field produces a magnetic field in the surrounding space, and vice-versa. These induced fields tend to form whirlpool patterns.

The most important consequence of the principle of induction is that there are no purely magnetic or purely electric waves. Disturbances in the electrical and magnetic fields propagate outward as combined magnetic and electric waves, with a well-defined relationship between their magnitudes and directions. These electromagnetic waves are what light is made of, but other forms of electromagnetic waves exist besides visible light, including radio waves, x-rays, and gamma rays.

Fields of force contain energy. The density of energy is proportional to the square of the magnitude of the field. In the case of static fields, we can calculate potential energy either using the previous definition in terms of mechanical work or by calculating the energy stored in the fields. If the fields are not static, the old method gives incorrect results and the new one must be used.
Problems

Key

✓ A computerized answer check is available online.
∫ A problem that requires calculus.
★ A difficult problem.

1 In an electrical storm, the cloud and the ground act like a parallel-plate capacitor, which typically charges up due to frictional electricity in collisions of ice particles in the cold upper atmosphere. Lightning occurs when the magnitude of the electric field builds up to a critical value, \( E_c \), at which air is ionized.
   (a) Treat the cloud as a flat square with sides of length \( L \). If it is at a height \( h \) above the ground, find the amount of energy released in the lightning strike.
   (b) Based on your answer from part a, which is more dangerous, a lightning strike from a high-altitude cloud or a low-altitude one?
   (c) Make an order-of-magnitude estimate of the energy released by a typical lightning bolt, assuming reasonable values for its size and altitude. \( E_c \) is about \( 10^6 \) V/m.

See problem 21 for a note on how recent research affects this estimate.

2 The neuron in the figure has been drawn fairly short, but some neurons in your spinal cord have tails (axons) up to a meter long. The inner and outer surfaces of the membrane act as the “plates” of a capacitor. (The fact that it has been rolled up into a cylinder has very little effect.) In order to function, the neuron must create a voltage difference \( V \) between the inner and outer surfaces of the membrane. Let the membrane’s thickness, radius, and length be \( t \), \( r \), and \( L \). (a) Calculate the energy that must be stored in the electric field for the neuron to do its job. (In real life, the membrane is made out of a substance called a dielectric, whose electrical properties increase the amount of energy that must be stored. For the sake of this analysis, ignore this fact.) [Hint: The volume of the membrane is essentially the same as if it was unrolled and flattened out.]
   (b) An organism’s evolutionary fitness should be better if it needs less energy to operate its nervous system. Based on your answer to part a, what would you expect evolution to do to the dimensions \( t \) and \( r \)? What other constraints would keep these evolutionary trends from going too far?

3 Consider two solenoids, one of which is smaller so that it can be put inside the other. Assume they are long enough so that each one only contributes significantly to the field inside itself, and the interior fields are nearly uniform. Consider the configuration where the small one is inside the big one with their currents circulating in the same direction, and a second configuration in which the currents circulate in opposite directions. Compare the energies of these configurations with the energy when the solenoids are far apart. Based
on this reasoning, which configuration is stable, and in which configuration will the little solenoid tend to get twisted around or spit out? [Hint: A stable system has low energy; energy would have to be added to change its configuration.]

4 The figure shows a nested pair of circular wire loops used to create magnetic fields. (The twisting of the leads is a practical trick for reducing the magnetic fields they contribute, so the fields are very nearly what we would expect for an ideal circular current loop.) The coordinate system below is to make it easier to discuss directions in space. One loop is in the $y-z$ plane, the other in the $x-y$ plane. Each of the loops has a radius of 1.0 cm, and carries 1.0 A in the direction indicated by the arrow.

(a) Using the equation in optional section 6.2, calculate the magnetic field that would be produced by one such loop, at its center.
(b) Describe the direction of the magnetic field that would be produced, at its center, by the loop in the $x-y$ plane alone.
(c) Do the same for the other loop.
(d) Calculate the magnitude of the magnetic field produced by the two loops in combination, at their common center. Describe its direction.

5 (a) Show that the quantity $\sqrt{4\pi k/\mu_0}$ has units of velocity.
(b) Calculate it numerically and show that it equals the speed of light.
(c) Prove that in an electromagnetic wave, half the energy is in the electric field and half in the magnetic field.

6 One model of the hydrogen atom has the electron circling around the proton at a speed of $2.2 \times 10^6$ m/s, in an orbit with a radius of 0.05 nm. (Although the electron and proton really orbit around their common center of mass, the center of mass is very close to the proton, since it is 2000 times more massive. For this problem, assume the proton is stationary.) In homework problem 9 on page 103, you calculated the electric current created.

(a) Now estimate the magnetic field created at the center of the atom by the electron. We are treating the circling electron as a current loop, even though it’s only a single particle.
(b) Does the proton experience a nonzero force from the electron’s magnetic field? Explain.
(c) Does the electron experience a magnetic field from the proton? Explain.
(d) Does the electron experience a magnetic field created by its own current? Explain.
(e) Is there an electric force acting between the proton and electron? If so, calculate it.
(f) Is there a gravitational force acting between the proton and electron? If so, calculate it.
(g) An inward force is required to keep the electron in its orbit –
otherwise it would obey Newton’s first law and go straight, leaving
the atom. Based on your answers to the previous parts, which force
or forces (electric, magnetic and gravitational) contributes signifi-
cantly to this inward force?

7  [You need to have read optional section 6.2 to do this prob-
lem.] Suppose a charged particle is moving through a region of space
in which there is an electric field perpendicular to its velocity vec-
tor, and also a magnetic field perpendicular to both the particle’s
velocity vector and the electric field. Show that there will be one
particular velocity at which the particle can be moving that results
in a total force of zero on it. Relate this velocity to the magnitudes
of the electric and magnetic fields. (Such an arrangement, called a
velocity filter, is one way of determining the speed of an unknown
particle.)

8  If you put four times more current through a solenoid, how
many times more energy is stored in its magnetic field? √

9  Suppose we are given a permanent magnet with a complicated,
asymmetric shape. Describe how a series of measurements with
a magnetic compass could be used to determine the strength and
direction of its magnetic field at some point of interest. Assume that
you are only able to see the direction to which the compass needle
settles; you cannot measure the torque acting on it. ⋆

10  Consider two solenoids, one of which is smaller so that it
can be put inside the other. Assume they are long enough to act
like ideal solenoids, so that each one only contributes significantly
to the field inside itself, and the interior fields are nearly uniform.
Consider the configuration where the small one is partly inside and
partly hanging out of the big one, with their currents circulating in
the same direction. Their axes are constrained to coincide.

(a) Find the magnetic potential energy as a function of the length
x of the part of the small solenoid that is inside the big one. (Your
equation will include other relevant variables describing the two
solenoids.)

(b) Based on your answer to part (a), find the force acting between
the solenoids.
11 Four long wires are arranged, as shown, so that their cross-section forms a square, with connections at the ends so that current flows through all four before exiting. Note that the current is to the right in the two back wires, but to the left in the front wires. If the dimensions of the cross-sectional square (height and front-to-back) are $b$, find the magnetic field (magnitude and direction) along the long central axis.

12 To do this problem, you need to understand how to do volume integrals in cylindrical and spherical coordinates. (a) Show that if you try to integrate the energy stored in the field of a long, straight wire, the resulting energy per unit length diverges both at $r \to 0$ and $r \to \infty$. Taken at face value, this would imply that a certain real-life process, the initiation of a current in a wire, would be impossible, because it would require changing from a state of zero magnetic energy to a state of infinite magnetic energy. (b) Explain why the infinities at $r \to 0$ and $r \to \infty$ don’t really happen in a realistic situation. (c) Show that the electric energy of a point charge diverges at $r \to 0$, but not at $r \to \infty$.

A remark regarding part (c): Nature does seem to supply us with particles that are charged and pointlike, e.g., the electron, but one could argue that the infinite energy is not really a problem, because an electron traveling around and doing things neither gains nor loses infinite energy; only an infinite change in potential energy would be physically troublesome. However, there are real-life processes that create and destroy pointlike charged particles, e.g., the annihilation of an electron and antielectron with the emission of two gamma rays. Physicists have, in fact, been struggling with infinities like this since about 1950, and the issue is far from resolved. Some theorists propose that apparently pointlike particles are actually not pointlike: close up, an electron might be like a little circular loop of string.

13 The purpose of this problem is to find the force experienced by a straight, current-carrying wire running perpendicular to a uniform magnetic field. (a) Let $A$ be the cross-sectional area of the wire, $n$ the number of free charged particles per unit volume, $q$ the charge per particle, and $v$ the average velocity of the particles. Show that the current is $I = Avnq$. (b) Show that the magnetic force per unit length is $AvnqB$. (c) Combining these results, show that the force
on the wire per unit length is equal to $IB$.  

14 Suppose two long, parallel wires are carrying current $I_1$ and $I_2$. The currents may be either in the same direction or in opposite directions. (a) Using the information from section 6.2, determine under what conditions the force is attractive, and under what conditions it is repulsive. Note that, because of the difficulties explored in problem 12, it’s possible to get yourself tied up in knots if you use the energy approach of section 6.5. (b) Starting from the result of problem 13, calculate the force per unit length.

15 The figure shows cross-sectional views of two cubical capacitors, and a cross-sectional view of the same two capacitors put together so that their interiors coincide. A capacitor with the plates close together has a nearly uniform electric field between the plates, and almost zero field outside; these capacitors don’t have their plates very close together compared to the dimensions of the plates, but for the purposes of this problem, assume that they still have approximately the kind of idealized field pattern shown in the figure. Each capacitor has an interior volume of 1.00 m$^3$, and is charged up to the point where its internal field is 1.00 V/m. (a) Calculate the energy stored in the electric field of each capacitor when they are separate. (b) Calculate the magnitude of the interior field when the two capacitors are put together in the manner shown. Ignore effects arising from the redistribution of each capacitor’s charge under the influence of the other capacitor. (c) Calculate the energy of the put-together configuration. Does assembling them like this release energy, consume energy, or neither?

16 Section 6.2 states the following rule:

For a positively charged particle, the direction of the $F$ vector is the one such that if you sight along it, the $B$ vector is clockwise from the $v$ vector.

Make a three-dimensional model of the three vectors using pencils or rolled-up pieces of paper to represent the vectors assembled with their tails together. Now write down every possible way in which the rule could be rewritten by scrambling up the three symbols $F$, $B$, and $v$. Referring to your model, which are correct and which are incorrect?

17 Prove that any two planar current loops with the same value of $IA$ will experience the same torque in a magnetic field, regardless of their shapes. In other words, the dipole moment of a current loop can be defined as $IA$, regardless of whether its shape is a square.
18 A Helmholtz coil is defined as a pair of identical circular coils separated by a distance, \( h \), equal to their radius, \( b \). (Each coil may have more than one turn of wire.) Current circulates in the same direction in each coil, so the fields tend to reinforce each other in the interior region. This configuration has the advantage of being fairly open, so that other apparatus can be easily placed inside and subjected to the field while remaining visible from the outside. The choice of \( h = b \) results in the most uniform possible field near the center. (a) Find the percentage drop in the field at the center of one coil, compared to the full strength at the center of the whole apparatus. (b) What value of \( h \) (not equal to \( b \)) would make this percentage difference equal to zero?

19 (a) In the photo of the vacuum tube apparatus in section 6.2, infer the direction of the magnetic field from the motion of the electron beam. (b) Based on your answer to a, find the direction of the currents in the coils. (c) What direction are the electrons in the coils going? (d) Are the currents in the coils repelling or attracting the currents consisting of the beam inside the tube? Compare with part a of problem 14.

20 In the photo of the vacuum tube apparatus in section 6.2, an approximately uniform magnetic field caused circular motion. Is there any other possibility besides a circle? What can happen in general? *

21 In problem 1, you estimated the energy released in a bolt of lightning, based on the energy stored in the electric field immediately before the lightning occurs. The assumption was that the field would build up to a certain value, which is what is necessary to ionize air. However, real-life measurements always seemed to show electric fields strengths roughly 10 times smaller than those required in that model. For a long time, it wasn’t clear whether the field measurements were wrong, or the model was wrong. Research carried out in 2003 seems to show that the model was wrong. It is now believed that the final triggering of the bolt of lightning comes from cosmic rays that enter the atmosphere and ionize some of the air. If the field is 10 times smaller than the value assumed in problem 1, what effect does this have on the final result of problem 1?

22 In section 6.2 I gave an equation for the magnetic field in the interior of a solenoid, but that equation doesn’t give the right answer near the mouths or on the outside. Although in general the computation of the field in these other regions is complicated, it is possible to find a precise, simple result for the field at the center of one of the mouths, using only symmetry and vector addition. What is it? *

Solution, p. 209