This is a complete video-based high school physics course that includes videos, labs, and hands-on learning. You can use it as your core high school physics curriculum, or as a college-level test prep course. Either way, you’ll find that this course will not only guide you through every step preparing for college and advanced placement exams in the field of physics, but also give you in hands-on lab practice so you have a full and complete education in physics. Includes text reading, exercises, lab worksheets, homework and answer keys.
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While you can do the entire course entirely on paper, it’s not really recommended since physics is based in real-world observations and experiments! Here’s the list of materials you need in order to complete all the experiments in this unit. Please note: you do not have to do ALL the experiments in the course to have an outstanding science education. Simply pick and choose the ones you have the interest, time and budget for.

- string – 3 feet long with a weight that can be tied to the end
- timer or stopwatch
- masking tape
- tongue-depressor size popsicle stick
- 3” x 1/4” rubber bands (3)
- index cards
- scissors
- hot glue gun
- rope (20 to 50 feet OR a jump rope)
- slinky
- AA batteries (4, cheap “dollar-store” carbon-zinc kind work great)
- AA battery case (4) (Radio Shack #270-408)
- Alligator clip leads (Radio Shack #278-1156)
- DC, 3V motor (2) (Radio Shack #273-223)
- popsicle sticks
Sound is a form of energy. Energy is the ability to move something over a distance against a force. So what is moving to make sound energy?

Molecules. Molecules are vibrating back and forth at fairly high rates of speed, creating waves. Energy moves from place to place by waves. Sound energy moves by longitudinal waves (the waves that are like a slinky). The molecules vibrate back and forth, crashing into the molecules next to them, causing them to vibrate, and so on and so forth. All sounds come from vibrations. In this unit, we will be taking a careful look at vibration, frequency, and resonance.

Be sure to take out a notebook and copy down each example problem right along with me so you take good notes as you go along. It’s a totally different experience when you are actively involved by writing down and working through each problem rather than passively sitting back and watching.
VIBRATIONS

In previous lessons we've learned that energy is the ability to do work, and that work is moving something a distance against a force. The concept of energy is fairly easy to see as far as lifting things or pushing things go. We are exerting energy to lift a box against the force of gravity. We are exerting energy to pedal our bike up a hill. But how does this energy stuff relate to light, electricity, or sound? What's moving against a force there?

With energy, what's happening is that outrageously tiny particles are moving back and forth outrageously tiny amounts, at outrageously high speeds. With light, you've got little photons moving, with electricity little electrons. With sound, you've got molecules moving back and forth.

This back and forth motion is called vibration and these vibrations make waves. When one particle moves back and forth it does work on another particle, which does work on another particle and so on. As these particles do work on one another, they cause a wave to move from one place to another.

Energy moves by waves or, in other words, waves are energy-mobiles! Before we get in over our heads talking about waves however, we need to spend some time on this vibration thing.

If you imagine a swing at the park, it's got a normal resting position (called an equilibrium position). That's when it's experiencing a balance of forces: the pull of gravity balances with the tension in the chain holding up the swing, and so forth. Everything adds up to zero, and stays at zero until a kid comes by.

When a kid starts pumping on the swing, the swing is no longer in equilibrium and the swing starts to move back and forth through an arc. This is a slow forced vibration that stops soon after the kid leaves the swing, and the kid needs to keep pumping back and forth to keep swinging at the same rate (or increase the swing rate). Eventually, the kid gets tired and stops pumping, and each repetition of the swing moving back and forth (vibration) is less and less (called damping). Eventually, the swing comes to a stop (equilibrium) until another energetic kid comes along.
DAMPING

Damping is when a spring, swing, or other vibrating object loses its energy over time. It means that without adding energy into the system, like pumping on a swing or hitting a drum head, the object will eventually come to its non-vibrating (equilibrium) position.

Imagine the kid on the swing again. Why does the kid move past the equilibrium point without stopping?

It’s actually because of Newton’s law of inertia. The swing continues on its path as long as there are balanced forces acting on it. As the swing moves past its equilibrium point, a restoring force acts on it to move it back to its original equilibrium position.

The difference between a vibration (like a kid swinging) and translational motion (like a kid on roller skates) is that the kid on skates can be permanently displaced from his starting position. The kid on a swing doesn’t move away from its original equilibrium position for long – it stops, turns around, and comes back due to the restoring force. An object that vibrates actually wiggles and jiggles around a fixed position (the equilibrium position).

Other examples of vibrating objects include tuning forks (which are like inverted swings), piano strings, car suspension systems, a weight on the end of a spring that bobs up and down, and drum heads. In each of these examples, when the object is disturbed (like hitting the tuning fork or drum head, or pulling the weight at the end of the spring), the object moves from its original resting position, stops and heads back toward its resting position and overshoots, stops and come back toward the resting position and the cycle repeats. The object moves along the same path over and over again, and if there was no friction or drag force (or other energy losses), the object would continue to move back and forth forever.

Did you notice that when you graph out the motion of the mass on a spring, it has a particular shape? The shape is described in math as a sinusoidal wave, or \( y = \sin(x) \). Did you notice how it is periodic and that you could also see the damping effect? The damping effect means that energy is being lost or dissipated. (Scientists wouldn’t say that it’s slowing down, because that indicates that the speed is decreasing, which isn’t the case when the mass stops and turns around to head back to the resting position. It is actually speeding up!)
FREQUENCY

The concept of frequency is very important to understanding energy. When it comes to electromagnetic waves it is frequency that determines whether the wave is radio, light, heat, microwave or more. It’s all the same type of energy, it’s the frequency that determines what that energy actually does. With sound energy the frequency determines the pitch of the sound.

As we move forward with energy, it is quite important that you know that all waves come from some sort of vibrating particle somewhere. The reason you can pick up a signal on your radio is because somewhere, maybe miles away, there is a particle vibrating at some ridiculous speed, creating a wave that moves across distances to finally vibrate the particles inside your radio’s antenna. It’s important to realize, however, that the particle does not move over that distance. The particle that started the wave back at the radio station is still there. It did not move to your radio it just vibrated at the antenna and started the wave.

Frequency is a measure of how many times something moves back and forth. A swing, a pendulum, a leg of a walking person all have a frequency. All those things start at one place, move, and come back to the same position that they started. This moving and coming back is one vibration. The faster something vibrates, the more frequency that something has.

Frequency is measured in Hertz. One Hertz (or Hz for short) is one vibration in one second. The Hertz is named after Heinrich Rudolf Hertz (1857-1894) a German physicist and professor. Hertz proved that electricity can be transmitted in electromagnetic waves, which travel at the speed of light and which possess many other properties of light. His experiments with these electromagnetic waves led to the development of the wireless telegraph and the radio.

A Hertz is a relatively slow vibration so there are also kilohertz (KHz), megahertz (MHz), and gigahertz (GHz). A kilohertz is 1000 Hz, a megahertz is 1,000,000 (a million) Hz, and a gigahertz is 100,000,000 (one thousand million) Hz.

Some examples of things that work at these frequencies are AM radio stations which broadcast at KHz, FM stations which broadcast at MHz, and microwaves which cook your food with GHz. If your radio is “crankin” tunes from radio station 750 AM, a part of your radio is vibrating at 750,000 times a second. If you’re “pumping wattage into your cottage” with WSCI at 94.2 on your radio dial, a part of your radio is vibrating at 94,200,000 times a second. If your radio happens to be green, then light is vibrating off your radio at 6 x 10^{14} Hz. That’s 6 with 14 zeros behind it or 600,000,000,000,000 vibrations in one second. That’s some serious vibes!

(By the way, if you can hear the sound coming out of your radio, your speakers are vibrating anywhere between 20 and 20,000 Hz. See how vibrations are important? They’re everywhere!) Let’s look more carefully at what those vibrations make, and that’s waves.
What is Frequency?

**Introduction:** When talking about sound, frequency tells us the pitch of the sound. A bird chirping is an example of a high frequency sound, while the rumble of an engine is a low frequency sound. But frequency has to do with more than just sound. Frequency is the rate at which something happens. All waves have frequencies. Pendulums also have frequencies.

In this experiment you will be adjusting the length of string of a pendulum until you get a pendulum that has a frequency of .5 Hz, 1 Hz and 2 Hz. Remember, a Hz is one vibration (or in this case swing back and forth) per second. So .5 Hz would be half a swing per second (swing one way but not back to the start). 1 Hz would be one full swing per second. Lastly, 2 Hz would be two swings per second. A swing is the same as a vibration so the pendulum must move away from where you dropped it and then swing back to where it began for it to be one full swing/vibration.

**Materials:**
- 3 foot long string
- A weight that can be tied to the end of the string
- A timer or stopwatch
- Meter stick
- Masking tape
- A table or chair
- A partner is helpful

**Procedure:**
1. Tie your weight (the official name of the weight on the end is bob. Personally I’ve always preferred the name Shirley, but Bob it is) to the end of the 3 foot string. If you’ve done the gravity lesson in the Mechanics set of lessons you’ll remember that the weight of the bob doesn’t matter. Gravity accelerates all things equally, so your pendulum will swing at the same speed no matter what the weight of the bob.
2. Tape the string to a table or chair or door jam. Make sure it can swing freely at about 3 feet of length.
3. I would recommend starting with 1 Hz. It tends to be the easiest to find. Then try .5 Hz and then 2 Hz.
4. The easiest way I’ve found to do this is to start the pendulum swinging and at the same time start the timer. Count how many swings you get in ten seconds.

5. Now, adjust the string. Make it longer or shorter and try again. When you get 10 swings in 10 seconds you got it! That’s one swing per second. You should be able to get quite close to one swing per second which is 1 Hz.

6. Now try to get .5 Hz. In this case you will get 5 swings in ten seconds when you find it. (A little hint, the string is pretty long here.)

7. Now speed things up a bit and see if you can get 2 Hz. Be prepared to count quick. That’s 2 swings a second or 20 swings in 10 seconds! (Another little hint, the string is quite short for this one.)

Did you get all three different frequency pendulums? It takes a while but my classes found it rather fun. You’ve created three different frequencies. 2 Hz being the fastest frequency. That was pretty fast right? Can you imagine something going at 10 Hz? 100 Hz? 1,000,000 Hz? I told you things were moving at outrageous speeds! The human ear can hear sounds ranging from about 20 Hz to 20,000 Hz. A 2 Hz sound wave would go totally unnoticed by us.

Now grab your meter stick. Align the beginning of the stick on the ground below the center of the pendulum swing (pictured below). Then pick a random spot on your string. Once you found a good spot, get ready to start your timer. Start swinging the pendulum, and once it’s reached a steady swing, record the distance the bob travels from its center point, and have your partner record it in the amplitude column on the table. Then, while the pendulum is still swinging, start the timer. Count how many swings the pendulum does in 10 seconds. Repeat this for two more random spots and record the number of swings per 10 seconds in the table on the next page:

![Pendulum](image)
Once you've filled out the table, let's convert swings per 10 seconds into Hertz. Take the number of swings you counted in 10 seconds, and divide that number by 10, and you'll have the number of swings per second (or Hz). Record these values in the frequency column.

Now let's plot one of your waves as a function of time. Choose one of your trials and plot the wave as a function of time. Include values along both axes. (Hint: mark intervals of your frequency along the time axis first).
One thing to note with pendulum motion is that there is no wave propagating through space. Basically, the wave is stationary, and has no physical wavelength. If you were to walk in a straight line while holding the swinging pendulum, then the motion of the pendulum would have a wavelength that would depend on how fast you were walking.

**Exercises:**

1. Imagine a large pendulum, with a 10 m long string connected to a 50 pound weight. What would be the frequency of the pendulum be if it swings 9.5 times in one minute?

2. Now imagine the weight is replaced with a 500 pound weight. What is the frequency now?

3. Now imagine the same pendulum is moving at 5 m/s in a straight line. What is the wavelength?
Answers:

1. .158 Hz

2. Still .158 Hz!

3. 31.5 meters
PERIOD

The period is the time it takes for one full cycle to complete itself and is measured in seconds per cycle. The frequency is the number of cycles that are made in a period of time, and is measured in cycles per second.
THE FREQUENCY AND PERIOD RELATIONSHIP

The frequency is the number of cycles that are made in a given period of time, like 10 swings in 5 seconds, and is measured in cycles per second. The period is the inverse of the frequency, given by this equation: $T = 1/\nu$. 
AMPLITUDE

Amplitude is how high or low the wave is from its original equilibrium position. (not vibrating).
How high can you get the swing to go? How far does the car system spring travel over that bump?
All these are the amplitude of the vibration.

Let’s try a sample problem for amplitude:
PENDULUMS

The restoring force slows down the object as it moves from its resting but speeds it up when it heads back to the resting position, and that’s what creates the vibration. We’re going to take a look at the forces in a pendulum from the point of view of Newton’s Laws.
There’s more than one way to solve physics problems... and by looking at the total mechanical energy of the system, you’ll be able to solve much more complicated pendulum problems with ease.
PENDULUM PERIOD

You'll need a pendulum for this experiment. A pendulum is really nothing more than a weight at the end of something that can swing back and forth. The easiest way to make one is to get a string and tape it to the edge of a table. (The string should be long enough so that it swings fairly close to the ground.) Tie a weight to the bottom of your string and you’ve got a pendulum.

Now, for YOUR part of the experiment we are going to change one of three different things, and only one thing changes at a time. First, we’ll change the length of string and measure the period. Then we’ll change the mass of the object, and then the angle that you start the pendulum from. With each trial you will be changing only ONE of those three things. (The rest of the variables will be constant.)

1. Make an observation. Play with the pendulum a bit and see how it behaves.
2. Make a hypothesis. How will the length of string effect the number of swings in 10 seconds? Will there be more, less, or no change in the number of swings as the string gets shorter.
3. Set a timer for 10 seconds or get someone who has a watch with a second hand to tell you when 10 seconds are up.
4. Now for the test. Pull the pendulum back as far as you’d like (the pendulum swings smoother if you don’t lift the weight higher than the top of the string).
5. Start the timer and let go of the weight at the same time.
6. Count the swings the pendulum makes in 10 seconds. This is your frequency in #cycles per 10 seconds.
7. Write down what you found (collect the data as shown in the video).
8. Do two more trials with the string at that same length.
9. Now change the changing variable. In other words, shorten the string. I would recommend shortening it at least an inch.
10. Redo steps 3 through 9, recording each time.
11. Continue shortening the string and doing trials until you get at least five different lengths of string.
12. Convert frequency to period by taking the inverse. Do this for each trial.

Now report your results. Take a look at your data and see if you find a trend. Do you get more swings as the string shortens, less swings, or does the length of the string matter? Something interesting to notice is that at a certain length you will get 10 swings in 10 seconds or a swing a second. This is why pendulums are used in grandfather clocks. They keep good time!
HOOKE’S LAW

How are pendulums like springs? They both vibrate, but how you model them on paper is a little different. Let’s take a look at how you handle springs and what their periodic nature looks like:
Waves are the way energy moves from place to place. Sound moves from a mouth to an ear by waves. Light moves from a light bulb to a book page to your eyes by waves. Waves are everywhere. As you sit there reading this, you are surrounded by radio waves, television waves, cell phone waves, light waves, sound waves and more. (If you happen to be reading this in a boat or a bathtub, you’re surrounded by water waves as well.) There are waves everywhere!

Do you remember where all waves come from? Vibrating particles. Waves come from vibrating particles and are made up of vibrating particles.

Here’s rule one when it comes to waves….the waves move, the particles don’t. The wave moves from place to place. The wave carries the energy from place to place. The particles however, stay put. Here’s a couple of examples to keep in mind.

If you’ve ever seen a crowd of people do the “wave” in the stands of a sporting event you may have noticed that the people only “vibrated” up and down. They did not move along the wave. The wave, however, moved through the stands.

Another example would be a duck floating on a wavy lake. The duck is moving up and down (vibrating) just like the water particles but he is not moving with the waves. The waves move but the particles don’t. When I talk to you, the vibrating air molecules that made the sound in my mouth do not travel across the room into your ears. (Which is especially handy if I’ve just eaten an onion sandwich!) The energy from my mouth is moved, by waves, across the room.

Why are waves energy-mobiles? Remember that energy is the ability to do work, and work is moving something a distance against a force. Can you tell me what is moving against a force in a wave?

If you said particles you’re right. Water particles, molecules, electrons, some sort of small particles are moving back and forth at potentially incredible speeds against a force. Each particle moving does work on another particle which gets it moving. That particle then does work on another particle which gets it moving, which then does work on another particle getting it moving, which then gets another moving and so on and so forth. Particles moving and doing work on other particles is energy and waves are how energy moves.

Physics is really nice to us here because, believe or not, with all the different forms of energy there are only two types of waves to remember; transverse and longitudinal. Neat, huh? That makes it pretty easy. So let’s talk about them. A transverse wave is a wave where the particle moves perpendicular to the medium. A longitudinal wave is where the particle moves parallel to the medium.

Let’s take a look at the different waves using a real jump rope:
A transverse wave has particles moving perpendicular to the direction that the wave is moving in. If the wave moves from left to right, you know it’s a transverse wave if the particles moves up and
down. This is the kind of wave that fans do in a stadium when they do the “wave”. Transverse waves always need a solid medium to travel through.

A longitudinal wave has particles that move parallel to the direction of the wave. Sound waves are longitudinal waves. Waves that travel through gases or liquids are longitudinal.

Earthquakes are both transverse and longitudinal waves along the solid parts of the Earth.

For both types of waves, the direction that energy is traveling in follows the wave, no matter which way that the individual particles are moving.

(Note: there are other forms of waves, like surface waves on the ocean that move in circular paths, and also torsional waves like when you rotate a slinky, but for the most part, there are just two main types of waves.)
MOTION OF WAVES

Here’s rule one when it comes to waves: the waves move, the particles don’t. The wave moves from place to place. The wave carries the energy from place to place. The particles however, stay put. Here’s a couple of examples to keep in mind.

If you’ve ever seen a crowd of people do the ‘wave’ in the stands of a sporting event you may have noticed that the people only vibrated up and down. They did not move along the wave. The wave, however, moved through the stands.

Another example would be a duck floating on a wavy lake. The duck is moving up and down (vibrating) just like the water particles but he is not moving with the waves. The waves move but the particles don’t. When I talk to you, the vibrating air molecules that made the sound in my mouth do not travel across the room into your ears. (Which is especially handy if I’ve just eaten an onion sandwich!) The energy from my mouth is moved, by waves, across the room.

A medium is the material that carries the wave. It isn’t the wave nor does it make the wave... it’s just the thing that carries the wave. When football fans in a stadium do “the wave”, the wave people are the medium that the wave travels through. For sound waves, air is the medium that sound travels through.
ENERGY OF A WAVE

Since the particles don't travel with the wave, what does a wave carry? Waves transport energy, not particles (or matter).
MECHANICAL AND ELECTROMAGNETIC WAVES

Some waves need a medium to travel through while others do not. Mechanical waves need a medium for the wave to travel through to transport energy. Ocean waves, jump ropes, pendulums, sound, and waves in a stadium are all examples of mechanical waves.

Electromagnetic waves do not need a medium to travel through. Light from the sun reaches up 93 million miles away by traveling though the vacuum of space because it’s an electromagnetic wave. The electromagnetic wave travels through the vacuum of space at the speed of light (299,792,458 m/s). These types of waves are made by vibrating charged particles, and we’re going to look at this more in depth in our next section on Light.

Matter waves are the ones you get to learn about when you study quantum physics, as they describe the way that matter (like a beam of electrons) under certain conditions acts like a wave. That’s way out of our scope here, but I want you to at least be aware that they exist.
The words *particle* and *wave* are two words you’ll see in nearly every area of physics, but they are actually very different from each other. A particle is a tiny concentration of something that can transmit energy, and a wave is a broad distribution of energy that fills the space it passes through. We’re going to look at particles in more depth later, and instead focus on understanding waves.

A wave traveling along a stretched string can have different shapes, but every wave will have a *frequency* and a *wavelength*. Wavelength refers to the repeating wave shape, and frequency refers to the oscillating source that make the wave in the first place. Waves are defined by a math equation that we’ll get to a little later. First, let’s take a look at the different parts of a wave. A wave number is the number of waves per unit length.

For longitudinal waves, a rarefaction is the spot where the wave is traveling and is the most “stretched out” (minimum wave density). The compression is the spot where the wave is most squished together. You can see this easily if you play with a slinky... the coils that are most spread out are a rarefaction point.
PERIOD, FREQUENCY, AMPLITUDE AND WAVELENGTH

A wave can have many different shapes, but it has a very specific frequency $\nu$ and wavelength $\lambda$. 
Humming birds are really fascinating, because they can beat their wings so fast! Here’s a quick way to calculate the frequency and period of a humming bird’s wings.
Kids love swings, and it’s amazingly simple to find the frequency and period of the swing. Here’s how...
ENERGY AND WAVE AMPLITUDE

How is the energy related to the wave frequency, amplitude and wavelength?
WAVE SPEED

Sound is a type of energy, and energy moves by waves. So sound moves from one place to another by waves; longitudinal waves to be more specific. So, how fast do sound waves travel? Well, that's a bit of a tricky question. The speed of the wave depends on what kind of stuff the wave is moving through. The more dense (thicker) the material, the faster sound can travel through it.

Remember that waves move because the particles bounce off one another? Well, the farther the particles are from one another, the longer it takes one particle to bounce off another. Think about a row of dominoes. If you put them all close together and push one over they all fall down pretty quick. If you spread them out a bit, the row falls much more slowly. Sound waves move the same way.
MORE ON WAVE SPEED

Sound moves faster in solid objects than it does in air because the molecules are very close together in a solid and very far apart in a gas. For example, sound travels at about 760 mph in air, 3300 mph in water, 11,400 mph in aluminum, and 27,000 mph in diamond!

The temperature of the material also makes a difference. The colder the material, the faster the sound. This is why sound seems to be louder or clearer in the winter or at night. The air is a little cooler and since it’s cooler, the molecules are a little more tightly packed.

The speed of a wave is based on the basic distance over time relationship. If you watch the crest of a wave, the speed is how fast the crest is observed to move a distance.
ECHOES

An echo is when a wave travels through one medium (like air) and then meets a different medium (like a cave wall). The sound wave bounces and reflects back to you.
Let’s do a couple of simpler sample problems, and then I’ll show you how to do problems that are more complex and involve higher level math. First, let’s take a look at the wings of a bird in flight...
Ocean waves travel on the surface of the water can be observed and measured. Let’s try one just before a storm...
SEASICK WAVES

Ever gotten sea sick? It's usually because the motion of what your body detects is different from what your eyes see. Let's take a look at how you can calculate the wave speed by watching two boats bobbing up and down (without getting sick).
WAVE SPEED ON TIGHT STRINGS

Waves traveling on a tight string, like a climbing rope, are dependent on only two things: the tension of the rope and a physical property of the rope (like what it’s made out of, the diameter, etc.).
WAVE EQUATIONS

Now it’s time for a little more math because the physics problems are going to get a little harder. The wave equation for transverse and longitudinal waves moving in the +x direction looks like this:

\[ y(x, t) = y_m \sin(\kappa x - \omega t) \]

where \( y_m \) is the amplitude of the wave, \( \kappa \) is the angular wave number, \( \omega \) is the angular frequency, and \( (\kappa x - \omega t) \) is the phase.

The wavelength \( L \) and the wave number \( k \) are found by the following equation:

\[ K = \frac{\kappa}{2\pi} = \frac{1}{\lambda} \]

where \( K \) is the number of waves per meter.

The period \( T \) and the frequency \( v \) are related to \( \omega \) by the following equation:

\[ v = \frac{\omega}{2\pi} = \frac{1}{T} \]

The main wave speed equation is given by:

\[ v = \frac{\omega}{\kappa} = \frac{\lambda}{T} = \lambda v \]

Note that the first letter \( v \) is for velocity, and the last letter in the equation is the Greek letter “nu” \( (\nu) \) for frequency.

The problems in the video involve using the first equation that relates the distance and time to find the amplitude, wave number, and frequency of a wave. (This is a typical problem that you’ll see in college level physics.)
WAVES ON A STRETCHED STRING

If a wave can travel through mediums like air, water, strings, rocks, etc., then it makes sense that as the wave moves through these mediums, the tiny particles that make up the medium will also vibrate. In order for this to happen, the medium has to have a way for energy (both potential and kinetic) to be stored, so the medium has both inertia and elasticity.

The wave equation for a stretched spring is:

\[ v = \sqrt{\frac{\tau}{\mu}} \]

where the first letter \( v \) is for velocity, \( \mu \) is the linear density of the string, and the tension is \( \tau \).

You can’t send a wave along a straight string without stretching the string. The tension in the string that does the stretching is the elasticity of the string that stores the potential energy as the wave passes through. The amount of tension in the rope will affect the wave speed. The wave speed doesn’t change if you change the frequency, however it will travel faster through a tighter rope.

This means that the speed of the wave depends on the medium, and not the wave itself. For example, waves travel faster through solid rock than they do through air because the particles in the medium are much closer together and can transmit the wave faster.

Here’s the power part of the video above explained in more detail:
BEHAVIOR OF WAVES

When a wave travels from one medium to another, like sound waves traveling in the air and then through a glass window pane, it crosses a boundary. Whether the wave continues to the new medium (and even how it goes through), or whether it bounces and reflects back, or a bit of both depends on the boundary.

Imagine a jump rope attached to a door handle. The last particle of the rope is fixed to the door handle, and doesn’t move at all. If you grab the free end of the rope and pull it taught, you have a nice, straight line. When you jerk the rope up, the pulse travels through the rope toward the door handle. Some of the energy carried by the pulse is reflected and comes back to you at the same speed and wavelength, but it’s upside down (called a reflected pulse) and not as large amplitude-wise. Some of the energy is also transmitted to the fixed end, causing the door handle to rattle and vibrate.

If you untie the rope from the door handle and instead tie it loosely to a pole (so it’s allowed to slide up and down easily) and repeat this experiment, you’ll find the pulse travels through the rope, turns around and reflects back right side up with the same speed and wavelength.

Now imagine untying the rope and instead attaching a rope denser, thicker rope to the end. The initial pulse travels toward the thicker rope, but two things happen when it hits the thicker rope: first, a reflected pulse (same speed and wavelength, but inverted) returns back to you, but also some of the energy goes into the thicker rope so a smaller, slower wave (but with the same frequency as the original pulse) will travel along the thicker rope. Waves travel fastest in the least dense medium, so the reflected wave travels faster than the transmitted wave. Even though the waves travel at different speeds in different mediums, they are all vibrating with the same frequency.

What would you expect to happen if a sound wave traveled from the denser rope to a less dense rope? The initial pulse (also called the incident pulse) travels through the denser medium, and when it hits the lesser dense rope, it undergoes partial transmission and partial reflection like this: the reflected pulse has the same speed and wavelength (and is right side up). Second, the wave transmitted to the less dense rope is right-side up, larger amplitude, and traveling faster than the reflected pulse.

The bottom line? Waves travel fastest in the least dense medium. Frequency doesn’t change when you cross a boundary. The wavelength is always greatest in the least dense rope. The amplitude is always greatest in the initial (incident) wave.

The reflected wave inverts when it moves from a less dense to a more dense rope due to Newton’s Laws of Motion. For the case when the rope is free to slide up and down on the pole, when the initial wave reaches the pole, the rope slides up and because of its inertia, it overshoots and exerts a reaction force on the string, and this reaction force sends a reflected wave back down the string (called a soft reflection). With the case of being fixed on a door handle, when the incident wave reaches the end, it exerts an upward force on the door handle, but Newton’s Third Law states that there’s an equal and opposite (reaction) force that the door handle exerts on the string, which generates an inverted pulse that travels back along the string (called a hard reflection).
WAVE REFLECTION

Straight waves are what happens when something moves back and forth in a medium like water. These are interesting when they hit a diagonal plane barrier, because when the incident wave reaches the barrier, the waves always reflect at the same angle that they approached the barrier with (called the Law of Reflection).

We'll look at how waves reflect off curved surfaces (like a parabolic mirror) when we get to our section on light waves.
WAVE REFRACTION

Waves bend when they go from one medium to another when the speed changes. It’s a really important topic in light (not so much with sound), because it’s how lenses, eyes, cameras, and telescopes work. The bending of sound waves happens naturally in the air above the earth when it’s warmer than the surface of the earth. The sound waves that travel through the warmer air are faster and the ones that travel through cooler air are slower. When the sound waves go from warmer to cooler air (less dense to more dense air), they become bent back down toward the surface.

So refraction can bend sound downward which in turn amplifies the sound by adding to the direct (original source) sound. If you’ve ever been near a cool lake, you’ve heard one of nature’s amplifiers!

But why does sound bend? You can imagine a toy car going from a wood floor to carpeting. One wheel hits the carpet first and slows down before the other, causing the toy to turn. The direction of the sound changes in addition to the speed. The slower speed must also shorten its wavelength since the frequency of the wave doesn’t change. Here’s the wave equation again from the previous section:

\[ v = \frac{\omega}{\kappa} = \frac{\lambda}{T} = \lambda \nu \]

Note the first \( v \) is velocity, and the last \( \nu \) is frequency in the equation.
DIFFRACTION

When waves pass around small (we’re talking small compared to the wavelength of the wave) objects, they diffract. People in the audience of a concert can hear really well if they are sitting right behind a pillar because the sound waves are large enough to bend around it (which is actually because of both diffraction and reflection effects). Diffraction helps sound bend around obstacles. You can sometimes hear conversations around corners because of diffraction.

Waves also diffract if they spread out after moving through small (again, smaller than wavelength-size) openings, like light going through a slit cut with a razor.

The amount a wave diffracts (bends) depends on the wavelength. Lower frequencies bend around objects better than higher frequencies. If you’ve ever watched a lightning and thunder storm, you know that there’s a lot more sound (like a sharp crack) when the lightning is closer (you hear both higher and lower frequencies) than when it’s more distant (mostly lower frequencies). Owls use low frequency sounds to transmit sounds further than the higher frequency bird twitters.
SUPERPOSITION

Often two (or more) waves travel through the same spot. If you’ve ever listened to an orchestra, you’re hearing the sounds from many different instruments all playing at the same time. If more than two boats are on the lake, their wakes churn up the water together. Here’s how we handle this in physics...

The principle of superposition states that when waves interfere with each other, the displacement at any location is the sum of the displacements of the waves at that location. Said another way, you simply add up the waves at the same spot to get the resulting amplitude.
INTERFERENCE

What if two waves of the same wavelength and amplitude travel in the same direction along a stretched string? What will the string look like? We know about the idea of superposition adding the waves together, but what does the string actually look like?

How waves interact with each other depends on whether the waves are in phase or not. If they are in-step (in phase) with each other, then it’s easy to add up to double the displacement (constructive interference). If they are completely out of step, then they cancel each other out (destructive interference).
When two waves have an increase in displacement when they interact, it’s constructive interference.
DESTRUCTIVE INTERFERENCE

Destructive interference happens when two waves have opposite displacements. The pulses don't destroy each other (as the name implies), but rather they cancel out the effect of each other when they interact with each other.

They don't have to cancel each other out completely to be destructive interference. They don't even have to have the same amplitudes. (And actually, when two waves meet, they don't even alter their path or alter the waveform itself after the interaction. They simply add or subtract when they interact, and then go on their merry way as they had been before the interaction when they're done. It's really quite amazing.)
WHICH KIND OF INTERFERENCE IS IT?

Let’s practice figuring out which interference is being observed when waves interact...
Imagine a police car on the side of the road with lights and sirens on full blast. You're also parked and you hear the same frequency (say 1,000 Hertz) of the siren. However, if you're driving at 75 mph toward the police car you're going to hear a higher frequency (1096 Hz), and if you're driving away at 75 mph, you're going to hear a lower frequency of 904 Hz. Why is that?

It has to do with a motion-related frequency change called the Doprller Effect. This effect was initially conceived by Johann Doppler in 1842 and later tested by Buys Ballot in Holland in 1845 with a locomotive drawing an open car with trumpeters. There's Doppler Effect for not only sound but also light including microwaves, radio waves, and visible light. Police use the Doppler Effect on their radar guns to track your speed. Astronomers use it to find the motions of stars, galaxies, and quasars... it's really an amazing tool in our scientific toolbox!
STANDING WAVES

When most people think of waves, they imagine something like an ocean wave... it moves through the water and isn’t confined to one area. Some waves don’t travel at all – they are called standing waves.

Picture two waves traveling in opposite directions. One is going to the right, the other going to the left. What happens when they combine? Using superposition, we can figure out what the interacting waves look like when they interfere with each other.

Did you notice how there are places along the string where the string is not moving at all? It’s at permanent rest... it’ doesn’t vibrate at all. Those places are called nodes. The places where the amplitude is greatest is called the antinodes.
NODES AND ANTINODES

One common misconception is the idea that nodes and antinodes are the same as the crest and trough of a wave. They’re not. A node is a place on the wave that is permanently at rest. An antinode is where the wave is at its maximum (it will travel through a large up and a large down displacement).

You’ll find standing waves when you look at columns of vibrating air or strings vibrating at resonance. This happens because of two things going on simultaneously: when the wave hits the barrier, instead of continuing on, it gets reflected back.

The reflected wave constructively interferes with the next incoming waves, and the overall effect is that you have places on the string that never move and places where it’s always at a maximum. This effect is what we call a standing wave.

Nodes and antinodes aren’t really part of the wave because a standing wave really isn’t a wave at all... it’s just a visual effect that looks like a wave that doesn’t move. You have to perfectly time the interference of two (or more) opposite-traveling waves to get this effect.
Standing waves are basically two waves traveling in opposite directions that constructively interfere with each other so it looks like the whole system is moving in simple harmonic motion.

Simple harmonic motion is when we used Hooke’s Law for the mass-spring system to figure out the displacement, period, or frequency of the system.

First harmonics have 2 nodes and 1 antinode. Second harmonics have 3 nodes and 2 antinodes, third harmonics have 4 nodes and 3 antinodes, and so on.
BUILD A STANDING WAVE MACHINE

Standing wave machines are fun to make because they are easy to build and amazing to watch! Here’s a simple one you can make on your own using the materials from the previous lessons in electricity.

Materials:

- AA batteries (4, cheap “dollar-store” carbon-zinc kind work great)
- AA battery case (4)
- Alligator clip leads
- DC, 3V motor (2)
- Hot glue gun (and glue sticks)
- Masking tape
- Popsicle sticks
- String
- Scissors

How many wavelengths can you see in yours? Where are the nodes and antinodes? Change the tension in the strong to get different harmonics to show up! (Note – if you know how to change the motor speed using a potentiometer, you can do that also!)

Also note – if you’re finding that the string wads into a tight ball after only a few seconds, it means that one of your motors is going the wrong direction. Stop both motors and switch the wires in the back of ONE of the motors to reverse the polarity (plus and minus) so the motor spins in the opposite direction. You want the one motor to wind up the string and the other to un-wind it at the same time. Eventually, since the motors spin at slightly different speeds, the string will get wound up and you’ll need a new piece of string.
It’s easy to calculate the frequency, period, wavelength and speed of waves at different harmonics.

For a tight string, there’s a handy equation that we can use to relate the length of the string to the wavelength based on the harmonic number like this:

\[ L = \left( \frac{n}{2} \right) \lambda \]

where \( L \) is the length of the string, and \( n \) is the harmonic number. Here’s how to use this equation to find everything we need about the wave:
HOMEWORK PROBLEMS WITH SOLUTIONS

On the following pages is the homework assignment for this unit. When you’ve completed all the videos from this unit, turn to the next page for the homework assignment. Do your best to work through as many problems as you can. When you finish, grade your own assignment so you can see how much you’ve learned and feel confident and proud of your achievement!

If there are any holes in your understanding, go back and watch the videos again to make sure you’re comfortable with the content before moving onto the next unit. Don’t worry too much about mistakes at this point. Just work through the problems again and be totally amazed at how much you’re learning.

If you’re scoring or keeping a grade-type of record for homework assignments, here’s my personal philosophy on using such a scoring mechanism for a course like this:

It’s more advantageous to assign a “pass” or “incomplete” score to yourself when scoring your homework assignment instead of a grade or “percent correct” score (like a 85%, or B) simply because students learn faster and more effectively when they build on their successes instead of focusing on their failures.

While working through the course, ask a friend or parent to point to three questions you solved correctly and ask you why or how you solved it.

Any problems you didn’t solve correctly simply mean that you’ll need to go back and work on them until you feel confident you could handle them when they pop up again in the future.
Student Worksheet for Waves

After you’ve worked through the sample problems in the videos, you can work out the problems below to practice doing this yourself. Answers are given on the last page.

Equations:

\[ y(x, t) = y_m \sin(kx - \omega t) \]

\[ v = \frac{\omega}{K} = \frac{\lambda}{T} = \lambda v \]

\[ K = \frac{\kappa}{2\pi} = \frac{1}{\lambda} \]

\[ v = \frac{\omega}{2\pi} = \frac{1}{T} \]

\[ \nu = \sqrt{\frac{T}{\mu}} \]

Practice Problems:

1. A traveling wave has a wavelength of 2.0 m and a velocity of 250 m/s. What are the frequency and period of the wave?

2. A traveling wave has a period of 0.75 milliseconds and a velocity of 175 m/s. What is the wavelength and frequency of the wave?
3. A traveling wave is made in a string. The time for one point to move from the minimum displacement to the maximum is 0.45 seconds and the wavelength is 0.25 m. What are the frequency, speed, and period of the wave?

4. A surfer rides a 10 m wave and reaches the dock 40 m away in 30 seconds. While surfing he waves at his friend who is riding a wave right behind him by 20 m. What is the speed, amplitude and wavelength of the wave?

5. A kid throws a ball into the pool and observes that 5 waves hit the edge of the pool in 10 seconds. He also notices that the first wave reaches the edge of the pool that is 7 meters away in 18 seconds. What are the wave speed, frequency, period, and wavelength?
6. Write the displacement equation for a sinusoidal wave that is traveling in the positive x-direction with frequency 100 Hz, speed 200 m/s, amplitude 0.020 mm, and phase constant π/4 rad?

7. The wave speed on a string is 100 m/s when the tension is 80 N. What tension will give you a speed of 150 m/s?

8. A 50 g string that is 2 meters long is under 70 N of tension. How fast will one wave travel across the string?
9. Find the speed of a transverse wave when the mass of the rope is 30 g, tension is 200 N, and the length is 1.3 meters.

10. Find the mass of rope when the speed of the transverse wave produced is 12 m/s, tension is 50 N and the length is 2.1 m.

11. Determine the tension in a rope when the mass is 70g, the speed is 6m/s and the length is 1.9 m.
12. You hold your ear to one end of a 140m copper pipe (Velocity of sound in copper = 3560 m/s) as your friend knocks on the other end. How much faster does the sound in the pipe reach you than the sound in the air?

13. Calculate how long it takes sound to travel through a 100 m long aluminum pipe. (speed of sound in aluminum is 6420m/s)

14. Write the displacement equation for a very long string with a linear density of 20.0g/m is stretched along the x-axis with a tension of 50N. At the origin the string is tied to a 100Hz automatic oscillating machine that vibrates the string to a max height of 5 cm above the origin at t=0.
15. A transverse wave is traveling through a string and is described by: \( y = (0.50 \ m) \sin [(1.5 \ m^{-1})x + (20 \ s^{-1})t] \).
What are the wave speed and the tension in the string when the linear density is 100g/m?

16. What are the three lowest frequencies for standing waves on a 100 g string that is 15m long and is stretched under 275 N of tension?

17. A 0.60m long string is fixed at both end and vibrates in the 1st harmonic with a wave speed of 300m/s. What is the wavelength and frequency of the wave?
18. A 1.5 m long bass string is fixed at both end and vibrating in the 1st harmonic. The speed of sound in air is 348 m/s and the frequency is 500 Hz. What is the speed of the waves in the wire and what is the wavelength of the sound produced?

19. Your friend is singing a single note at 300 Hz while running towards you at 15 m/s. What frequency do you hear when the speed of sound is 340 m/s?

20. A police car speeds past you at 25 m/s emitting a siren of 1450 Hz. While driving 19 m/s, what frequency do you hear?
Advanced Physics

Student Worksheet for Waves

After you’ve worked through the sample problems in the videos, you can work out the problems below to practice doing this yourself. Answers are given on the last page.

Equations:

\[ y(x,t) = y_m \sin(kx - \omega t) \]

\[ k = \frac{\lambda}{2\pi} = \frac{1}{\lambda} \]

\[ v = \frac{\omega}{k} = \frac{1}{\lambda} \]

\[ v = \frac{\lambda}{T} = \lambda \sqrt{\frac{T}{\mu}} \]

Practice Problems:

1. A traveling wave has a wavelength of 2.0 m and a velocity of 250 m/s. What are the frequency and period of the wave?

\[ \lambda = 2.0 \text{ m} \]

\[ v = 250 \text{ m/s} \]

\[ \frac{\lambda}{\lambda} = \frac{2}{2} = 1 \]

\[ v = \frac{250 \text{ m/s}}{2} = 125 \text{ Hz} \]

\[ \frac{T}{v} = \frac{2}{250 \text{ m/s}} = 0.008 \text{ s} \]

2. A traveling wave has a period of 0.75 milliseconds and a velocity of 175 m/s. What is the wavelength and frequency of the wave?

\[ T = 0.00075 \text{ s} \]

\[ v = 175 \text{ m/s} \]

\[ \frac{1}{v} = \frac{1}{0.00075} = 1333.33 \text{ Hz} \]

\[ \lambda(T) = \frac{v}{T} = \frac{175 \text{ m/s}}{0.00075} = 0.131 \text{ m} \]

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3. A travelling wave is made in a string. The time for one point to move from the minimum displacement to the maximum is 0.45 seconds and the wavelength is 0.25 m. What are the frequency, speed, and period of the wave?

\[ T = 0.45 \text{ s} \]
\[ \frac{1}{T} = \nu \]
\[ \frac{1}{0.45} = 1.11 \text{ Hz} \]
\[ \lambda = 0.25 \text{ m} \]
\[ \nu = \frac{\lambda}{T} = \frac{0.25}{0.009} = 0.278 \text{ m/s} \]

4. A surfer rides a 10 m wave and reaches the dock 40 m away in 30 seconds. While surfing he waves at his friend who is riding a wave right behind him by 20 m. What is the speed, amplitude and wavelength of the wave?

\[ A = 10 \text{ m} \]
\[ \nu = \frac{\text{distance}}{\text{time}} = \frac{40 \text{ m}}{30 \text{ s}} = 1.33 \text{ m/s} \]
\[ \lambda = 20 \text{ m} \]

5. A kid throws a ball into the pool and observes that 5 waves hit the edge of the pool in 10 seconds. He also notices that the first wave reaches the edge of the pool that is 7 meters away in 18 seconds. What are the wave speed, frequency, period, and wavelength?

\[ \nu = \frac{\text{oscillations}}{\text{time}} = \frac{5}{10 \text{ sec}} = 0.5 \text{ Hz} \]
\[ \frac{1}{\nu} = T = \frac{1}{0.5} = 2.0 \text{ sec} \]
\[ \nu = \frac{\text{distance}}{\text{time}} = \frac{7 \text{ m}}{18 \text{ s}} = 0.389 \text{ m/s} \]
\[ \lambda = \frac{\nu}{\nu} = \frac{(0.389 \text{ m/s})}{(0.5 \text{ Hz})} = 0.778 \text{ m} \]
6. Write the displacement equation for a sinusoidal wave that is traveling in the positive x-direction with frequency 100 Hz, speed 200 m/s, amplitude 0.020 mm, and phase constant π/4 rad?

\[ v = 100 \text{ Hz} \]
\[ \nu = 200 \text{ m/s} \]
\[ A = 2.0 \times 10^{-5} \text{ m} \]
\[ \phi = \frac{\pi}{4} \]

\[ \lambda = \frac{\nu}{v} = 2.0 \text{ m} \]
\[ \kappa = \frac{2 \pi}{\lambda} = \pi \]

\[ (2.0 \times 10^{-5} \text{ m}) \sin(\pi x - \pi(100)t + \frac{\pi}{4}) \]

7. The wave speed on a string is 100 m/s when the tension is 80 N. What tension will give you a speed of 150 m/s?

\[ v = 100 \text{ m/s} \]
\[ T = 80 \text{ N} \]
\[ \frac{v}{\sqrt{T}} = \mu \]

\[ \frac{80 \text{ N}}{(100 \text{ m/s})^2} = 0.0080 \text{ kg/m} \]

\[ 150 \text{ m/s} = \sqrt{\frac{T}{0.0080}} \]

\[ T = 180 \text{ N} \]

8. A 50 g string that is 2 meters long is under 70 N of tension. How fast will one wave travel across the string?

\[ \mu = \frac{\text{mass}}{\text{distance}} \]

\[ \frac{0.050 \text{ kg}}{2.0 \text{ m}} = 0.025 \text{ kg/m} \]

\[ v = \sqrt{\frac{T}{\mu}} \]

\[ \sqrt{\frac{70 \text{ N}}{0.025}} = 52.9 \text{ m/s} \]

\[ t = \frac{\text{distance}}{\text{velocity}} \]

\[ \frac{2.0 \text{ m}}{52.9 \text{ m/s}} = 0.0378 \text{ s} \]
9. Find the speed of a transverse wave when the mass of the rope is 30 g, tension is 200 N, and the length is 1.3 meters.

\[
\mu = \frac{0.030 \text{ kg}}{(1.3 \text{ m})} = 0.0231 \text{ kg/m}
\]

\[
U = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{200 \text{ N}}{0.0231 \text{ kg/m}}} = 93.1 \text{ m/s}
\]

10. Find the mass of rope when the speed of the transverse wave produced is 12 m/s, tension is 50 N and the length is 2.1 m.

\[
U = 12 \text{ m/s} \quad T = 50 \text{ N} \quad \ell = 2.1 \text{ m}
\]

\[
\frac{T}{U^2} = \mu
\]

\[
50 \text{ N} \quad \frac{50 \text{ N}}{(12 \text{ m/s})^2} = 0.372 \text{ kg/m}
\]

\[
\mu = \frac{m}{\ell} = \frac{(0.372 \text{ kg})(2.1 \text{ m})}{m} = 0.729 \text{ kg}
\]

11. Determine the tension in a rope when the mass is 70 g, the speed is 6 m/s and the length is 1.9 m.

\[
U = 6.0 \text{ m/s} \quad m = 0.070 \text{ kg} \quad \ell = 1.9 \text{ m}
\]

\[
\mu = \frac{0.070 \text{ kg}}{(1.9 \text{ m})} = 0.0368 \text{ kg/m}
\]

\[
U^2(\mu) = T
\]

\[
(6.0)^2(0.0368) = 1.33 \text{ N}
\]
12. You hold your ear to one end of a 140m copper pipe (velocity of sound in copper = 3560 m/s) as your friend knocks on the other end. How much faster does the sound in the pipe reach you than the sound in the air?

\[ v_{\text{sound in air}} = \frac{140\, \text{m}}{\text{343 m/s}} = 0.408 \, \text{sec} \]

\[ v_{\text{copper}} = \frac{140\, \text{m}}{\text{5000 m/s}} = 0.028 \, \text{sec} \]

\[ \text{Time in air} - \text{Time in copper} = 0.380 \, \text{sec} \]

13. Calculate how long it takes sound to travel through a 100 m long aluminum pipe. (speed of sound in aluminum is 6420 m/s)

\[ T = \frac{\text{distance}}{\text{wave speed}} \]

\[ \frac{100\, \text{m}}{6420 \, \text{m/s}} = 0.016 \, \text{sec} \]

14. Write the displacement equation for a very long string with a linear density of 20.0g/m is stretched along the x-axis with a tension of 50N. At the origin the string is tied to an automatic oscillating machine that vibrates the string to a max height of 5 cm above the origin at t=0.

\[ \mu = 0.02 \, \text{kg/m} \]

\[ T = 50 \, \text{N} \]

\[ v = 100 \, \text{Hz} \]

\[ A = 0.05 \, \text{m} \]

\[ n \, \text{at origin} \]

\[ \phi_0 = \frac{\pi}{2} \]

\[ \nu = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{50 \, \text{N}}{0.02 \, \text{kg/m}}} = 50 \, \text{m/s} \]

\[ \omega = 2\pi (v) = 200\pi \, \text{rad/s} \]

\[ \lambda = \frac{v}{\nu} = \frac{50 \, \text{m/s}}{100 \, \text{Hz}} = 0.5 \, \text{m} \]

\[ k = \frac{2\pi}{\lambda} = \pi \]

\[ (0.05 \, \text{m}) \sin (\pi x - 200\pi t + \frac{\pi}{2}) \]
15. A transverse wave is traveling through a string and is described by: \( y = (0.50 \text{ m}) \sin[(1.5 \text{ m}^{-1})x + (20 \text{ s}^{-1})t] \). What are the wave speed and the tension in the string when the linear density is 100g/m?

\[
\begin{align*}
\lambda &= \frac{2\pi}{k} = \frac{2\pi}{1.5} = 4.189 \text{ m} \\
\omega &= 20 \text{ s}^{-1} = \nu \left(\frac{2\pi}{1.5}\right) \rightarrow \nu = \frac{10}{\pi} \text{ Hz} \\
\nu &= \sqrt{\frac{T}{\mu}} \\
\nu^2 \mu &= T \\
(13.3)^2 \times (0.100) &= 17.7 \text{ N}
\end{align*}
\]

16. What are the three lowest frequencies for standing waves on a 100 g string that is 15m long and is stretched under 275 N of tension?

\[
\begin{align*}

m &= 0.100 \text{ kg} \\
 \ell &= 15 \text{ m} \\
 T &= 275 \text{ N} \\

\mu &= \frac{0.100 \text{ kg}}{15 \text{ m}} = 0.00667 \text{ kg/m} \\

\nu &= \sqrt{\frac{T}{\mu}} = \sqrt{\frac{275 \text{ N}}{0.00667 \text{ kg/m}}} = 203.1 \text{ m/s} \\

V &= \frac{203.1 \text{ m/s}}{15 \text{ m}} \\
V_m &= \frac{\nu}{2\ell} \text{ (Harmonics)} \\

V_m &= 6.77 \text{ Hz}, 13.54 \text{ Hz}, 20.31 \text{ Hz}
\end{align*}
\]

17. A 0.60m long string is fixed at both end and vibrates in the 1st harmonic with a wave speed of 300m/s. What is the wavelength and frequency of the wave?

\[
\begin{align*}
\ell &= 0.60 \text{ m} \\
\nu &= 300 \text{ m/s} \\
V &= \frac{\nu}{2\ell} = \frac{300 \text{ m/s}}{0.60 \text{ m}} = 250 \text{ Hz} \\
\ell &= \frac{2\lambda}{1} \\
\lambda &= 2 \times (0.60 \text{ m}) \\
\lambda &= 1.2 \text{ m}
\end{align*}
\]
18. A 1.5 m long bass string is fixed at both end and vibrating in the 1st harmonic. The speed of sound in air is 348 m/s and the frequency is 500 Hz. What is the speed of the waves in the wire and what is the wavelength of the sound produced?

\[ \ell = 1.5 \text{ m} \]

Harmonic \( n = 1 \)

\[ V_{\text{air}} = 348 \text{ m/s} \]

\[ V = 500 \text{ Hz} \]

\[ \frac{V}{2} = \frac{V_{\text{air}} - V}{2} \]

\[ V = \frac{V_{\text{air}}}{2} \]

\[ \lambda = \frac{348 \text{ m/s}}{500 \text{ Hz}} = 0.696 \text{ m} \]

19. Your friend is singing a single note at 300 Hz while running towards you at 15 m/s. What frequency do you hear when the speed of sound is 340 m/s?

\[ V_{+} = \frac{V_{0}}{1 - \frac{V_{s}}{V}} \]

\[ V_{0} = \frac{300 \text{ Hz}}{1 - \frac{15 \text{ m/s}}{340 \text{ m/s}}} = 313.8 \text{ Hz} \]

20. A police car speeds past you at 25 m/s emitting a siren of 1450 Hz. While driving 19 m/s, what frequency do you hear?

\[ V_{\text{car}} = \frac{(V_{\text{source}} + V_{\text{sound}})}{(V_{\text{object}} + V_{\text{sound}})} V_{\text{source}} \]

\[ V = \frac{(25 \text{ m/s} + 343 \text{ m/s})}{(19 \text{ m/s} - 343 \text{ m/s})} 1450 \text{ Hz} = 1640.9 \text{ Hz} \]
**ADVANCED PLACEMENT PHYSICS 1 EQUATIONS, EFFECTIVE 2015**

### CONSTANTS AND CONVERSION FACTORS

<table>
<thead>
<tr>
<th>Physical Quantity</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proton mass, $m_p$</td>
<td>$m_p = 1.67 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Neutron mass, $m_n$</td>
<td>$m_n = 1.67 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Electron mass, $m_e$</td>
<td>$m_e = 9.11 \times 10^{-31}$ kg</td>
</tr>
<tr>
<td>Speed of light, $c$</td>
<td>$c = 3.00 \times 10^8$ m/s</td>
</tr>
<tr>
<td>Electron charge magnitude, $e$</td>
<td>$e = 1.60 \times 10^{-19}$ C</td>
</tr>
<tr>
<td>Coulomb’s law constant, $k$</td>
<td>$k = \frac{1}{4\pi\varepsilon_0} = 9.0 \times 10^9$ N.m$^2$/C$^2$</td>
</tr>
<tr>
<td>Universal gravitational constant, $G$</td>
<td>$G = 6.67 \times 10^{-11}$ m$^3$/kg.s$^2$</td>
</tr>
<tr>
<td>Acceleration due to gravity at Earth’s surface, $g$</td>
<td>$g = 9.8$ m/s$^2$</td>
</tr>
</tbody>
</table>

#### UNIT SYMBOLS

<table>
<thead>
<tr>
<th>Unit</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>meter</td>
<td>m</td>
</tr>
<tr>
<td>kelvin</td>
<td>K</td>
</tr>
<tr>
<td>watt</td>
<td>W</td>
</tr>
<tr>
<td>degree Celsius</td>
<td>ºC</td>
</tr>
<tr>
<td>kilogram</td>
<td>kg</td>
</tr>
<tr>
<td>hertz</td>
<td>Hz</td>
</tr>
<tr>
<td>coulomb</td>
<td>C</td>
</tr>
<tr>
<td>second</td>
<td>s</td>
</tr>
<tr>
<td>newton</td>
<td>N</td>
</tr>
<tr>
<td>volt</td>
<td>V</td>
</tr>
<tr>
<td>ampere</td>
<td>A</td>
</tr>
<tr>
<td>joule</td>
<td>J</td>
</tr>
<tr>
<td>ohm</td>
<td>Ω</td>
</tr>
</tbody>
</table>

### PREFIXES

<table>
<thead>
<tr>
<th>Factor</th>
<th>Prefix</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{12}$</td>
<td>tera</td>
<td>T</td>
</tr>
<tr>
<td>$10^9$</td>
<td>giga</td>
<td>G</td>
</tr>
<tr>
<td>$10^6$</td>
<td>mega</td>
<td>M</td>
</tr>
<tr>
<td>$10^3$</td>
<td>kilo</td>
<td>k</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>centi</td>
<td>c</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>milli</td>
<td>m</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>micro</td>
<td>µ</td>
</tr>
<tr>
<td>$10^{-9}$</td>
<td>nano</td>
<td>n</td>
</tr>
<tr>
<td>$10^{-12}$</td>
<td>pico</td>
<td>p</td>
</tr>
</tbody>
</table>

### VALUES OF TRIGONOMETRIC FUNCTIONS FOR COMMON ANGLES

<table>
<thead>
<tr>
<th>$	heta$</th>
<th>0º</th>
<th>30º</th>
<th>45º</th>
<th>60º</th>
<th>90º</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin$	heta$</td>
<td>0</td>
<td>$1/2$</td>
<td>$\sqrt{3}/2$</td>
<td>$\sqrt{3}/2$</td>
<td>1</td>
</tr>
<tr>
<td>cos$	heta$</td>
<td>1</td>
<td>$\sqrt{3}/2$</td>
<td>$4/5$</td>
<td>$\sqrt{3}/2$</td>
<td>3/5</td>
</tr>
<tr>
<td>tan$	heta$</td>
<td>0</td>
<td>$\sqrt{3}/3$</td>
<td>3/4</td>
<td>1</td>
<td>$4/3$</td>
</tr>
</tbody>
</table>

The following conventions are used in this exam.

I. The frame of reference of any problem is assumed to be inertial unless otherwise stated.

II. Assume air resistance is negligible unless otherwise stated.

III. In all situations, positive work is defined as work done on a system.

IV. The direction of current is conventional current: the direction in which positive charge would drift.

V. Assume all batteries and meters are ideal unless otherwise stated.
### MECHANICS

\[ v_x = v_{x0} + a_x t \]
\[ x = x_0 + v_{x0}t + \frac{1}{2} a_x t^2 \]
\[ v_x^2 = v_{x0}^2 + 2a_x (x - x_0) \]
\[ \ddot{a} = \sum \vec{F} = \vec{F}_{\text{net}} \]
\[ |\vec{F}| \leq \mu |\vec{F}_N| \]
\[ a_c = \frac{v^2}{r} \]
\[ \vec{p} = m\vec{\nu} \]
\[ \Delta \vec{p} = \vec{F} \Delta t \]
\[ K = \frac{1}{2}mv^2 \]
\[ \Delta E = W = F_i d = F d \cos \theta \]
\[ P = \frac{\Delta E}{\Delta t} \]
\[ \theta = \theta_0 + \omega_0 t + \frac{1}{2} \omega t^2 \]
\[ \omega = \omega_0 + \omega t \]
\[ x = A \cos(2\pi f t) \]
\[ \ddot{a} = \sum \frac{\vec{F}}{I} = \frac{\vec{F}_{\text{net}}}{I} \]
\[ \Delta U_g = mg \Delta y \]
\[ T = \frac{2\pi}{\omega} = \frac{1}{f} \]
\[ L = I\omega \]
\[ \Delta L = \tau \Delta t \]
\[ K = \frac{1}{2} I \omega^2 \]
\[ |\vec{F}_g| = k|\vec{x}| \]
\[ U_s = \frac{1}{2} kx^2 \]
\[ \rho = \frac{m}{V} \]
\[ U_G = -\frac{Gm_1m_2}{r} \]

### ELECTRICITY

\[ |\vec{F}_E| = k \frac{|q_1q_2|}{r^2} \]
\[ A = \text{area} \]
\[ F = \text{force} \]
\[ I = \text{current} \]
\[ \ell = \text{length} \]
\[ P = \text{power} \]
\[ q = \text{charge} \]
\[ R = \frac{P}{I} \]
\[ r = \text{separation} \]
\[ t = \text{time} \]
\[ V = \text{electric potential} \]
\[ \rho = \text{resistivity} \]
\[ R_s = \sum R_i \]
\[ \frac{1}{R_p} = \sum \frac{1}{R_i} \]

### WAVES

\[ \lambda = \frac{v}{f} \]
\[ \nu = \text{speed} \]
\[ \lambda = \text{wavelength} \]

### GEOMETRY AND TRIGONOMETRY

**Rectangle**
\[ A = bh \]
\[ C = \text{circumference} \]
\[ V = \text{volume} \]

**Triangle**
\[ A = \frac{1}{2}bh \]
\[ S = \text{surface area} \]
\[ b = \text{base} \]
\[ h = \text{height} \]
\[ \ell = \text{length} \]
\[ w = \text{width} \]
\[ r = \text{radius} \]

**Circle**
\[ A = \pi r^2 \]
\[ C = 2\pi r \]

**Rectangular solid**
\[ V = \ell wh \]

**Right triangle**
\[ c^2 = a^2 + b^2 \]
\[ \sin \theta = \frac{a}{c} \]
\[ \cos \theta = \frac{b}{c} \]
\[ \tan \theta = \frac{a}{b} \]
## ADVANCED PLACEMENT PHYSICS 2 EQUATIONS, EFFECTIVE 2015

### CONSTANTS AND CONVERSION FACTORS

<table>
<thead>
<tr>
<th>Constants and Factors</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proton mass, $m_p$</td>
<td>$1.67 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Neutron mass, $m_n$</td>
<td>$1.67 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Electron mass, $m_e$</td>
<td>$9.11 \times 10^{-31}$ kg</td>
</tr>
<tr>
<td>Avogadro’s number, $N_0$</td>
<td>$6.02 \times 10^{23}$ mol$^{-1}$</td>
</tr>
<tr>
<td>Universal gas constant, $R$</td>
<td>$8.31$ J/(mol·K)</td>
</tr>
<tr>
<td>Boltzmann’s constant, $k_B$</td>
<td>$1.38 \times 10^{-23}$ J/K</td>
</tr>
<tr>
<td>Electron charge magnitude, $e$</td>
<td>$1.60 \times 10^{-19}$ C</td>
</tr>
<tr>
<td>1 electron volt, 1 eV</td>
<td>$1.60 \times 10^{-19}$ J</td>
</tr>
<tr>
<td>Speed of light, $c$</td>
<td>$3.00 \times 10^8$ m/s</td>
</tr>
<tr>
<td>Universal gravitational constant, $G$</td>
<td>$6.67 \times 10^{-11}$ m$^3$/kg·s$^2$</td>
</tr>
<tr>
<td>Acceleration due to gravity at Earth’s surface, $g$</td>
<td>$9.8$ m/s$^2$</td>
</tr>
<tr>
<td>1 unified atomic mass unit, $u$</td>
<td>$1.66 \times 10^{-27}$ kg</td>
</tr>
<tr>
<td>Planck’s constant, $h$</td>
<td>$6.63 \times 10^{-34}$ J·s</td>
</tr>
<tr>
<td>$hc$</td>
<td>$1.99 \times 10^{-25}$ J·m</td>
</tr>
<tr>
<td>Vacuum permittivity, $\varepsilon_0$</td>
<td>$8.85 \times 10^{-12}$ C$^2$/N·m$^2$</td>
</tr>
<tr>
<td>Coulomb’s law constant, $k = 1/(4\pi\varepsilon_0)$</td>
<td>$9.0 \times 10^9$ N·m$^2$/C$^2$</td>
</tr>
<tr>
<td>Vacuum permeability, $\mu_0$</td>
<td>$4\pi \times 10^{-7}$ (T·m)/A</td>
</tr>
<tr>
<td>Magnetic constant, $k' = \mu_0/4\pi$</td>
<td>$1 \times 10^{-7}$ (T·m)/A</td>
</tr>
<tr>
<td>1 atmosphere pressure, $1$ atm</td>
<td>$1.0 \times 10^5$ N/m$^2$</td>
</tr>
</tbody>
</table>

### UNIT SYMBOLS

<table>
<thead>
<tr>
<th>Unit</th>
<th>Symbol</th>
<th>Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Meter, m</td>
<td>m</td>
<td>1</td>
</tr>
<tr>
<td>Kilogram, kg</td>
<td>kg</td>
<td>$10^3$</td>
</tr>
<tr>
<td>Second, s</td>
<td>s</td>
<td>$10^9$</td>
</tr>
<tr>
<td>Ampere, A</td>
<td>A</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Coulomb, C</td>
<td>C</td>
<td>$10^9$</td>
</tr>
<tr>
<td>Joule, J</td>
<td>J</td>
<td>$10^{-9}$</td>
</tr>
<tr>
<td>Farad, F</td>
<td>F</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>Newton, N</td>
<td>N</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>Ohm, Ω</td>
<td>Ω</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Volt, V</td>
<td>V</td>
<td>$10^9$</td>
</tr>
<tr>
<td>Hertz, Hz</td>
<td>Hz</td>
<td>$10^3$</td>
</tr>
<tr>
<td>Hertz, Hz</td>
<td>Hz</td>
<td>$10^3$</td>
</tr>
<tr>
<td>Tesla, T</td>
<td>T</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Coulomb, C</td>
<td>C</td>
<td>$10^9$</td>
</tr>
<tr>
<td>Joule, J</td>
<td>J</td>
<td>$10^{-9}$</td>
</tr>
<tr>
<td>Farad, F</td>
<td>F</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>Newton, N</td>
<td>N</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>Ohm, Ω</td>
<td>Ω</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Volt, V</td>
<td>V</td>
<td>$10^9$</td>
</tr>
</tbody>
</table>

### VALUES OF TRIGONOMETRIC FUNCTIONS FOR COMMON ANGLES

<table>
<thead>
<tr>
<th>θ</th>
<th>0°</th>
<th>30°</th>
<th>37°</th>
<th>45°</th>
<th>53°</th>
<th>60°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>sinθ</td>
<td>0</td>
<td>1/2</td>
<td>3/5</td>
<td>√2/2</td>
<td>4/5</td>
<td>√3/2</td>
<td>1</td>
</tr>
<tr>
<td>cosθ</td>
<td>1</td>
<td>√3/2</td>
<td>4/5</td>
<td>√2/2</td>
<td>3/5</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>tanθ</td>
<td>0</td>
<td>√3/3</td>
<td>3/4</td>
<td>1</td>
<td>4/3</td>
<td>√3</td>
<td>∞</td>
</tr>
</tbody>
</table>

The following conventions are used in this exam.

I. The frame of reference of any problem is assumed to be inertial unless otherwise stated.

II. In all situations, positive work is defined as work done on a system.

III. The direction of current is conventional current: the direction in which positive charge would drift.

IV. Assume all batteries and meters are ideal unless otherwise stated.

V. Assume edge effects for the electric field of a parallel plate capacitor unless otherwise stated.

VI. For any isolated electrically charged object, the electric potential is defined as zero at infinite distance from the charged object.
### MECHANICS

\[ v_x = v_{x0} + a_x t \]
\[ x = x_0 + v_{x0} t + \frac{1}{2} a_x t^2 \]
\[ v_x^2 = v_{x0}^2 + 2a_x (x - x_0) \]
\[ \ddot{a} = \frac{\sum \ddot{F}_m}{m} = \ddot{F}_{net} \]
\[ |\ddot{F}_f| \leq \mu |\ddot{F}_n| \]
\[ a_c = \frac{v_e^2}{r} \]
\[ \ddot{p} = m \ddot{v} \]
\[ \Delta \ddot{p} = \ddot{F} \Delta t \]
\[ K = \frac{1}{2} m v^2 \]
\[ \Delta E = W = Fd = Fd \cos \theta \]
\[ P = \frac{\Delta E}{\Delta t} \]
\[ \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \]
\[ \omega = \omega_0 + \alpha t \]
\[ x = A \cos(\omega t) = A \cos(2\pi ft) \]
\[ x_{cm} = \sum \frac{m_i x_i}{m} \]
\[ T = \frac{2\pi}{\omega} = \frac{1}{f} \]
\[ \ddot{a} = \frac{\sum \ddot{F}_m}{I} = \ddot{F}_{net} \]
\[ T_s = 2\pi \sqrt{\frac{m}{k}} \]
\[ \tau = r_F = r \vec{F} \sin \theta \]
\[ L = I \omega \]
\[ \Delta L = \tau \Delta t \]
\[ K = \frac{1}{2} I \omega^2 \]
\[ |\ddot{F}_s| = k |\dddot{x}| \]
\[ U_G = -\frac{G m_1 m_2}{r} \]

### ELECTRICITY AND MAGNETISM

\[ |\ddot{F}_E| = \frac{1}{4\pi \varepsilon_0} \frac{|q_1 q_2|}{r^2} \]
\[ \ddot{E} = \ddot{F}_E \]
\[ |\ddot{E}| = \frac{1}{4\pi \varepsilon_0} \frac{|q|}{r^2} \]
\[ \ddot{V} = \frac{1}{\varepsilon_0} \frac{q}{r} \]
\[ \Delta U_E = q \Delta V \]
\[ V = \frac{q}{4\pi \varepsilon_0} r \]
\[ |\ddot{\vec{E}}| = \left| \frac{\Delta V}{\Delta r} \right| \]
\[ \Delta V = \frac{Q}{C} \]
\[ U_C = \frac{1}{2} Q \Delta V = \frac{1}{2} C (\Delta V)^2 \]
\[ \vec{F}_M = q \ddot{v} \times \vec{B} \]
\[ \vec{F}_M = I \ddot{\ell} \times \vec{B} \]
\[ R_s = \sum \frac{1}{R_i} \]
\[ \frac{1}{R_p} = \sum \frac{1}{R_i} \]
\[ \vec{E} = -\frac{\Delta \Phi_B}{\Delta t} \]
\[ B = \frac{\mu_0 I}{2\pi r} \]
\[ \phi_B = \vec{B} \cdot \vec{A} \]
\[ C_p = \sum_C C_i \]
\[ \frac{1}{C_s} = \sum \frac{1}{C_i} \]
### Fluid Mechanics and Thermal Physics

\[ \rho = \frac{m}{V} \]  
\[ P = \frac{F}{A} \]  
\[ F = P_0 + \rho gh \]  
\[ F_b = \rho V g \]  
\[ A_1 v_1 = A_2 v_2 \]  
\[ P_1 + \rho g y_1 + \frac{1}{2} \rho v_1^2 = P_2 + \rho g y_2 + \frac{1}{2} \rho v_2^2 \]  
\[ Q = \frac{kA \Delta T}{L} \]  
\[ PV = nRT = N k_B T \]  
\[ K = \frac{3}{2} k_B T \]  
\[ W = -P \Delta V \]  
\[ \Delta U = Q + W \]  

### Waves and Optics

\[ \lambda = \frac{v}{f} \]  
\[ n = \frac{c}{v} \]  
\[ n_1 \sin \theta_1 = n_2 \sin \theta_2 \]  
\[ \frac{1}{s_i} + \frac{1}{s_o} = \frac{1}{f} \]  
\[ |M| = \frac{h_i}{h_o} = \frac{s_i}{s_o} \]  
\[ \Delta L = m \lambda \]  
\[ d \sin \theta = m \lambda \]  

### Modern Physics

\[ E = hf \]  
\[ K_{\text{max}} = hf - \phi \]  
\[ \lambda = \frac{h}{p} \]  
\[ E = mc^2 \]  

### Geometry and Trigonometry

**Rectangle**  
\[ A = bh \]  
\[ C = \text{circumference} \]  
\[ V = \text{volume} \]  

**Triangle**  
\[ A = \frac{1}{2} bh \]  
\[ b = \text{base} \]  
\[ h = \text{height} \]  
\[ \ell = \text{length} \]  
\[ w = \text{width} \]  
\[ r = \text{radius} \]  

**Circle**  
\[ A = \pi r^2 \]  
\[ C = 2\pi r \]  

**Rectangular solid**  
\[ V = \ell wh \]  

**Right triangle**  
\[ c^2 = a^2 + b^2 \]  
\[ \sin \theta = \frac{a}{c} \]  
\[ \cos \theta = \frac{b}{c} \]  
\[ \tan \theta = \frac{a}{b} \]  

**Sphere**  
\[ V = \frac{4}{3} \pi r^3 \]  
\[ S = 4\pi r^2 \]