Explorations with a Paper Circle

Mathematical relationships can be found and explored in many situations. In honor of Pi Day, which occurs on March 14, this issue of *Student Math Notes* explores the circle. In the United States we write March 14, 2007, as 3/14/2007. The 3/14 suggests 3.14, which represents the first few digits of pi. If you drew circles of all different sizes, you would find that the circumference divided by the radius is the same value. This surprising discovery traces back to the Babylonians about 4000 years ago. Computers have been used to calculate the value of pi for hundreds of thousands of digits.

A parachute company wants to determine the location for a parachute landing target. The goal is to locate a landing target that is as far away as possible from all trees. One way to accomplish this task is by using an aerial photograph of the landing area. On the photograph the parachute company draws the largest possible circle that has no trees in the interior. The actual target for the landing area is the center of that circle. Once the company determines the largest possible circle in the area, it locates the center of the circle—by tracing the circle, cutting out the newly traced circle, and folding intersecting diameters. This procedure is related to the first task of this exploration.

Defining a Circle

A **circle** is the set of all points in a plane that are equidistant from a point called the center. Find a circular object about the size of a salad plate to trace (do not use a compass to draw the circle). Trace and cut out the circle. (Make a few extra circles in case you have to start over.) A circle is two dimensional, while the paper circle is three dimensional; technically, it is a disk. For the purposes of this investigation, we will ignore its depth and refer to the disk as a circle.

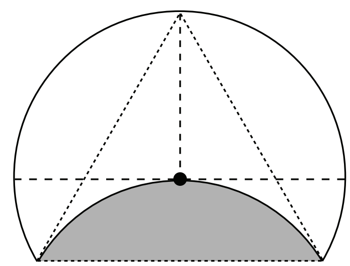
The first task is to find the center of the paper circle. The center of the circle is the **midpoint**, or middle, of the diameter. The **diameter** is the line segment that contains the center of the circle and whose endpoints lie on the circle—that is, the diameter is the longest distance across the circle.

1. Why does folding a circle in half locate the diameter of the circle?
2. What is the fewest number of folds necessary to determine the center of the circle?

Fold the circle to find the center.

3. How many other ways can you find to fold the circle while using the same number of folds used to determine the center?

The **radius** of a circle is a segment from the center of the circle to any point on the circle; the length of a circle's radius is half the length of its diameter. Open the circle and identify one radius by drawing a light line along part of one of the folds. After drawing the radius, fold up the part of the circle along that line so that the edge of the circle is at the center of the circle, as shown below. This fold is a **chord** of the circle, a line segment connecting any two points on the circle (the diameter is also a chord). It is also the **perpendicular bisector** of the radius. The perpendicular bisector is the line that intersects the midpoint of another line (the radius, in this case) at a right angle.



Perpendicular bisector

Developing the Area Formula for a Circle

You can relate the formula for the area of a circle to the formula for the area of a parallelogram: $A = l \times w$. Begin by dividing a circle into smaller and smaller sections. The photograph of Montreal's Olympic Stadium provides a nice image of these sections.

The Editorial Panel wishes to thank Alfinio Flores, Arizona State University, Tempe, AZ, Troy P. Regis, University of Missouri, Columbia, MO, Margaret Tent, Altamont School, Birmingham, AL, and William Speer, University of Nevada, Las Vegas, NV, for some of the ideas used in this issue of Student Math Notes.

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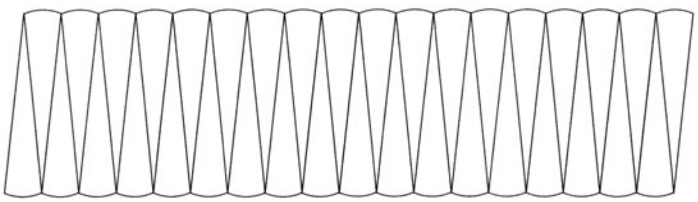
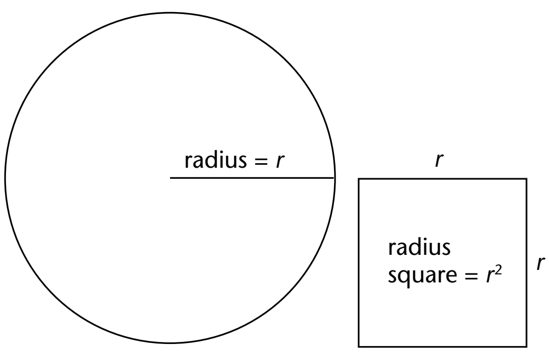
Then position the sections next to one another, alternating their direction, to form a parallelogram.



4. If the radius is r, what is the circumference of the circle?
5. What is the width of this parallelogram? What is the length of this parallelogram? What is the area of this parallelogram?
6. How does the area of this parallelogram relate to the area of the circle?
7. What can we conclude is the formula for finding the area of a circle?

You can also use a square to discover the formula for the area of a circle. Take an 8×10-inch sheet of grid paper and fold it in half. On one side, use a compass to draw a circle with radius r. On the other side, draw four $r \times r$ squares. The $r \times r$ square is called a **radius square**.

8. How many radius squares do you think will fit into the circle?

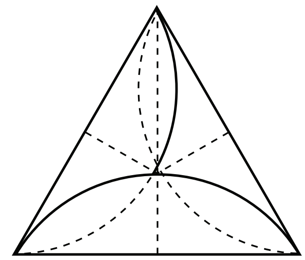


Color the four squares with different colors. Cut out the circle. Cut out the first square and fit it entirely inside the circle. Cut out the second square and fit it inside the circle without overlapping the previous color. You will have to cut parts of the second square so that they fit inside the circle without extending beyond it. Use all of the second square before using the third square. Continue with the third and fourth squares in the same manner. Save the remainder of the fourth square and estimate how much of the fourth square you were able to fit. Using grid paper can help you make more accurate estimations.

9. How many radius squares fit into your circle?
10. Compare your answer with your classmates' answers. How do they compare?

Exploring Areas of Other Polygons

Hold the circle so that the folded section is toward you. Make two more folds (toward you) to complete the triangle that has the folded perpendicular bisector as one side. The folded circle should now look like this:

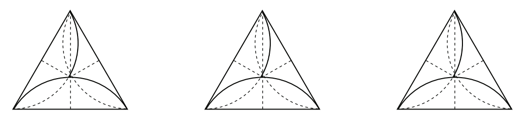


11. What kind of triangle is formed? Justify your answer.

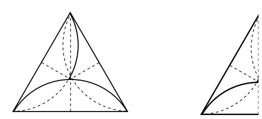
For the purpose of this exploration, identify this triangle as the **square unit of area**. It is called a square unit (even though it is a triangle) because it is used as a unit of area measure. All other areas in this activity will be based on this triangle as the square unit of area.

12. Using the triangle above to represent a square unit of area, describe the total area pictured below.

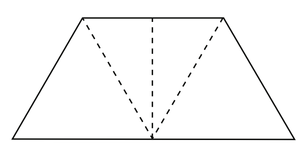
a.



b.



Turn the triangle so that the folds are facing away from you. Now fold one of the vertices of the triangle toward you to the midpoint of the opposite side. The midpoint of the opposite side can be found by folding the side of the triangle opposite the vertex in half. This will form a **trapezoid**—a quadrilateral (or four-sided polygon) with one pair of parallel sides. We will name this Trapezoid A.



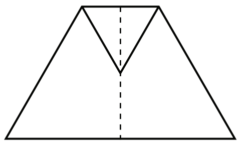
Trapezoid A

13. What special type of trapezoid is Trapezoid A? Explain.

To determine the area of this trapezoid, compare it with the square unit (the large triangle you folded earlier). Trapezoid A is less than 1 square unit because it is smaller than the triangle.

14. What fraction of our square unit does Trapezoid A represent? (Hint: Trapezoid A can be divided into three congruent triangles. How does the area of those triangles compare with the area of the larger triangle?)

Return the circle to the unit triangle. Now make an isosceles trapezoid by folding a vertex of the triangle to the center of the original circle. Name this Trapezoid B.

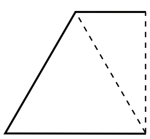


Trapezoid B

15. Using the area of the triangle as a square unit, determine the area of Trapezoid B.
16. What is the difference in area between Trapezoid A and Trapezoid B?

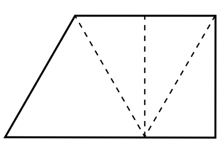
Return to Trapezoid A. An infinite number of possible non–isosceles trapezoids can be formed by folding Trapezoid A once. We will explore two of these possibilities.

Fold Trapezoid A in half along the location of the dotted vertical line in the illustration of Trapezoid A. Call this Trapezoid C.

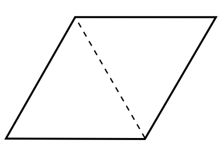


Trapezoid C

17. What is the area of Trapezoid C?
18. Fold Trapezoid A so that one of the vertices of the long side coincides with the midpoint of that side. What is the area of the resulting trapezoid?



Unfold the circle to return to Trapezoid A. Make a single fold to create a parallelogram, as shown below. This is a special parallelogram called a **rhombus**.



19. What makes this shape a rhombus?
20. Using the original triangle as the square unit of area, determine the area of this rhombus.

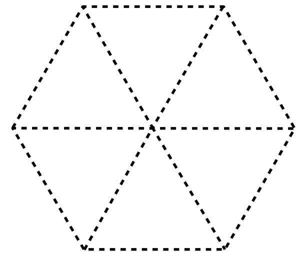
Make a single fold in the rhombus to create an equilateral triangle.

21. Using the original triangle as the square unit of area, determine the area of this triangle.

Let the triangle unfold somewhat so that it forms a **tetrahedron**, or triangular pyramid.

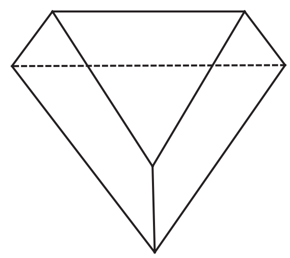
22. Using the original triangle as the square unit of area, determine the surface area of the tetrahedron.

Unfold the tetrahedron to return to the original triangle. Fold each vertex of the triangle to the center of the original circle, as shown below.



23. What is the name of this polygon?
24. Given our defined square unit of area, what is the area of this polygon?

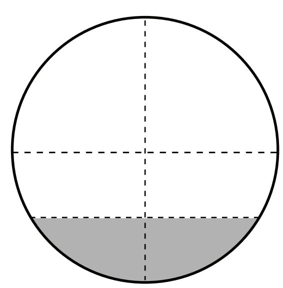
Let this polygon unfold slightly and tuck the "flaps" from the new folds into each other to form a **truncated tetrahedron**.



25. Using the original triangle as the square unit of area, determine the surface area of the truncated tetrahedron.

Can You …

- determine the length of the chord formed by the perpendicular bisector of the radius?
- determine the area of the arc segment designated by the shaded region below?



- determine how the unit triangle might be used to determine the area of the folded region above?

Did You Know That …

- the approximation for π has not always been 3.14? The ancient Babylonians (2000 B.C.) used 3 1/8, or 3.125, and the early Greeks used $\sqrt{10}$, or 3.162.
- Pi Day is often celebrated at 1:59 p.m. to recognize the six-digit approximation : 3.14159?
- some people also celebrate a Pi Approximation Day? This day can fall on several dates:

  July 22, or 22/7, a format used by many countries to write dates

  April 26, when the distance of the Earth's orbit divided by the time it has traveled so far is equal to π

  November 10, the 314th day of the year

  December 21 at 1:13 p.m., the 355th day of the year, in celebration of the Chinese approximation 355/113

Mathematical Content

Circles, triangles, trapezoids, area relationships, surface area, fractions

Resources

Flores, Alfinio, and Troy P. Regis. "How Many Times Does a Radius Square Fit into the Circle?" *Mathematics Teaching in the Middle School* 8 (March 2003): 363–68.

Sobel, M. A., and E. M. Maletsky. *Teaching Mathematics: A sourcebook of Aids, Activities, and Strategies.* 3d ed. Needham Heights, MA: Allyn and Bacon, 1999.

Tent, Margaret W. "Circles and the Number π." *Mathematics Teaching in the Middle School* 6 (April 2001): 452–57.

www.education2000.com/demo/demo/botchtml/areacirc.htm

The following sites contain information about Pi Day activities and more:

- Pi Day on the "Math with Mr. Herte" Web site: mathwithmrherte.com/pi_day.htm
- Pi pages on the Internet: joyofpi.com/pilinks.html
- The pi trivia game: eveander.com/trivia
- The world of pi: members.aol.com/loosetooth/pi.html
- A history of pi: www-groups.dcs.st-and.ac.uk:80/~history/HistTopics/Pi_through_the_ages.html
- Pi Day sites from the Math Forum: mathforum.com/t2t/faq/faq.pi.html
- Discovering pi: www.eduref.org/cgi-bin/printlessons.cgi/Virtual/Lessons/Mathematics/Geometry/GEO0001.html

Answers

1. By folding the circle in half, you are locating the longest segment by the line of symmetry. Folding the circle in half locates the line that passes through the center of the circle.
2. 2
3. There are an infinite number of ways, all going through the center of the circle.
4. $C = 2\pi r$
5. In this parallelogram, the width is r and the length is $(1/2)(2\pi r)$, so the area would be $A = \pi r^2$.
6. The area of the parallelogram is the same as the area of the circle.
7. The formula for the area of a circle is $A = \pi r^2$.
8. Answers will vary but should be between 2 and 4.
9. a little more than 3
10. All answers should be very close, regardless of the size of the circle and the radius square.
11. Equilateral. All three sides are the same length (congruent) because they are all perpendicular bisectors of a radius.
12. *a.* 3 square units *b.* 1.5 square units (or 1 1/2 square units)
13. It is an **isosceles trapezoid** (a quadrilateral with one pair of parallel sides and two congruent sides that are not parallel).
14. 3/4 square unit
15. 8/9 square unit
16. 5/36 square unit; 8/9 – 3/4 = 32/36 – 27/36
17. 1/2 × 3/4 = 3/8 square unit
18. 5/6 × 3/4 = 5/8 square unit
19. It is a quadrilateral with all sides the same length (congruent).
20. 2/4 (or 1/2) square unit
21. 1/4 square unit
22. 1 square unit
23. It is a **hexagon** (a polygon with six sides).
24. 6/9 (or 2/3) square unit
25. 7/9 square unit

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